Systems of Springs

Systems of springs can be used to model all kind of complex behavior, including the vibrations of molecules in crystals or in chemical bonds. We want to begin here with just two spring with no damping, with just one-dimensional motion, and work our way up to more complex models from there.

Example 1. Consider the model shown below. Write an equation that models the coupled spring-mass system.
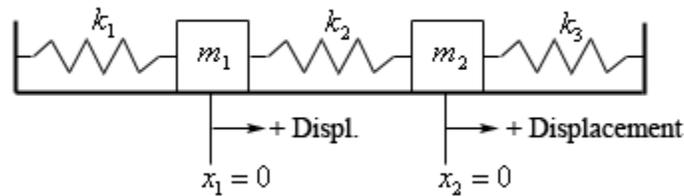
We want to find all the forces on each mass. We will do this by equating forces to get the signs correct. Then we can rearrange our equations to put the system in standard form if needed.

Recall that a general force model for acceleration $F = ma = mx''$ is equal to all other forces on the system. Here, the forces being applied are from Hooke's Law: $F = -kx$. The negative is because the force is applied in the opposite direction of motion. Because springs connect the masses, when one mass moved, this also creates a force on the other mass, so we will need to take that into account as well.

Let's consider m_1 and its position given by x_1. The acceleration here will be given by $F = m_1 x_1''$. To find the forces on the mass from the springs, first we find the forces from the motion of m_1 itself. Imagine everything else in the system is fixed: If m_1 moves to the right, a force is applied from spring #1, which is now stretched, pulling the mass back to the left: $-k_1 x_1$; and there is a second force from spring #2, which is now compressed, pushing the mass back to the left as well: $-k_2 x_1$. Now, fix m_1 and imagine moving m_2 to the right. This stretches spring #2 and applies a force pulling m_1 in the same direction as m_2: $+k_2 x_2$. The third spring does not apply force directly to m_1.

Adding these spring forces on the right side of the equation to the acceleration term on the left side, we get

$$m_1 x_1'' = -k_1 x_1 - k_2 x_1 + k_2 x_2$$

Applying the same reasoning to the second mass, the acceleration is $m_2 x_2''$. The forces on m_2 when m_2 is moved to the right are $-k_2 x_2$ and $-k_3 x_2$, and the force on m_2 from moving m_1 to the right is $+k_2 x_1$. This results in the equation

$$m_2 x_2'' = k_2 x_1 - k_2 x_2 - k_3 x_2$$

Solving for $\vec{x}''$

$$\vec{x}'' = \begin{bmatrix} \dfrac{-k_1 - k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & \dfrac{-k_2 - k_3}{m_2} \end{bmatrix}$$

If we have values for the masses and spring constants, we can insert them at the start, or at this point in the equation, in order to solve the system.

We can create a more realistic scenario by adding damping to the system.

Example 2. Consider the system below. Write the equation that models the system.
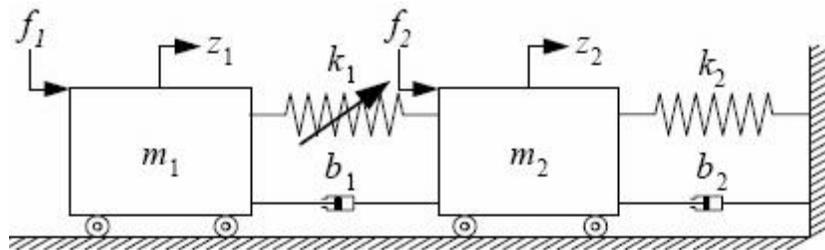
Recall that damping forces are given by $F = -\gamma v = -\gamma x'$ and these forces go on the same side of the equation as the spring forces. (In this diagram, the damping is given by b_1, b_2.)

In addition to $F = m_1 x_1''$, there are the spring forces from motion of m_1: $- k_1 x_1$ and spring forces from motion of m_2: $+ k_1 x_2$, and there is damping for each of these motions (with the same signs) from the damping between the two masses: $-b_1 x_1'$ and $+b_1 x_2'$. This results in the equation

$$m_1 x_1'' = -b_1 x_1' + b_1 x_2' - k_1 x_1 + k_1 x_2$$

The second mass has forces acting on it from both springs due to motion of the second mass: $-k_1 x_2 - k_2 x_2$ but also forces from the motion of the first mass: $+k_1 x_1$. There are also damping forces from motion of the second mass: $-b_1 x_2' - b_2 x_2'$, and damping forces from motion of the first mass: $+b_1 x_1'$. Both match the signs of the spring forces. This results in the equation

$$m_2 x_2'' = b_1 x_1' - b_1 x_2' - b_2 x_2' + k_1 x_1 - k_1 x_2 - k_2 x_2$$

In order to write this in matrix form, we'll need to make a substitution: let $x_1' = x_3, x_2' = x_4$ and $x_3 = x_1'', x_4' = x_2''$:

$$x_1' = x_3$$
$$x_2' = x_4$$
$$m_1 x_3' = -b_1 x_3 + b_1 x_4 - k_1 x_1 + k_1 x_2$$
$$m_2 x_4' = b_1 x_3 - b_1 x_4 - b_2 x_4 + k_1 x_1 - k_1 x_2 - k_2 x_2$$

Dividing by the masses and ordering the new terms in order, we obtain the equation

$$\vec{\dot{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{b_1}{m_1} & \dfrac{b_1}{m_1} \\ \dfrac{k_1}{m_2} & \dfrac{-k_1-k_2}{m_2} & \dfrac{b_1}{m_2} & \dfrac{-b_1-b_2}{m_2} \end{bmatrix} \vec{x}$$

We can combine springs with masses in other ways as well. We can combine them in series, or in parallel, depending on how we want to add the spring forces. The graphic below shows how the spring forces get added. Here the connection point B is assumed to be massless.
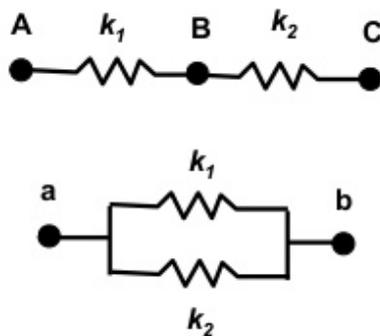
• Equivalent Spring

$$\text{series: } k_{AC} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

$$\text{parallel: } k_{ab} = k_1 + k_2$$

Example 3. Write the equation of the spring and mass model to the right.

The right side of the equation is given by mx''. All the masses are acting in the same direction on this single mass, so even though it is a system of springs, there is just one equation. The spring above the mass sum together normally to affect the motion of the mass: $-(k_1 + k_2)x$, while the ones below it, combine as the reciprocal of the sum of reciprocals: $-\left(\dfrac{1}{\frac{1}{k_3}+\frac{1}{k_4}}\right)x$. Resulting in the equation

$$mx'' = -(k_1 + k_2)x - \left(\frac{1}{\dfrac{1}{k_3} + \dfrac{1}{k_4}}\right)x$$
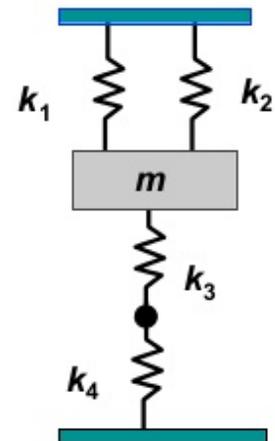
Or rearranging to put in the standard form for a single equation:

$$x'' + \left[\frac{k_1 + k_2}{m} + \left(\frac{1}{\dfrac{m}{k_3} + \dfrac{m}{k_4}}\right)\right]x = 0$$

This technique can be applied to individual masses in more complicated systems as we did with multiple springs and dampers in previous examples.

Practice problems. For each diagram below, construct the system of equations to model the setup using the values provided for each one.

1. $k_1 = k_2 = 1$,
   $m_1 = m_2 = 1$
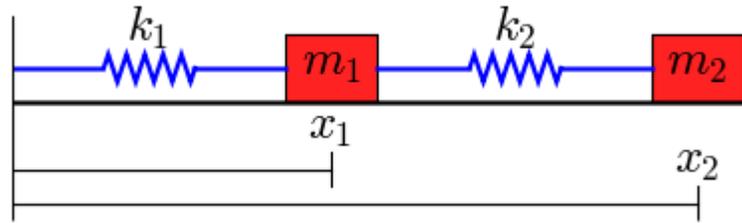
2. $\gamma = 1$
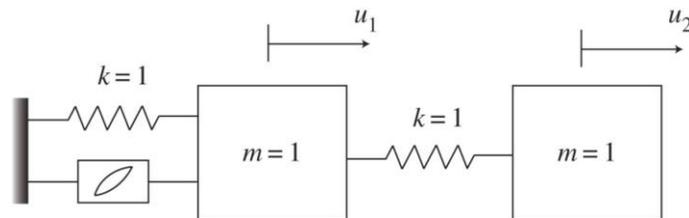
3. $k_1 = 4, k_2 = 2,\ c_1 = 3, c_2 = 1, m_1 = 8, m_2 = 3$
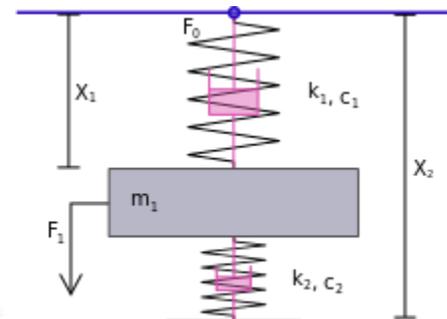
4. $k_1 = 10, k_2 = 3, k_3 = 7, m_1 = m_2 = 1$
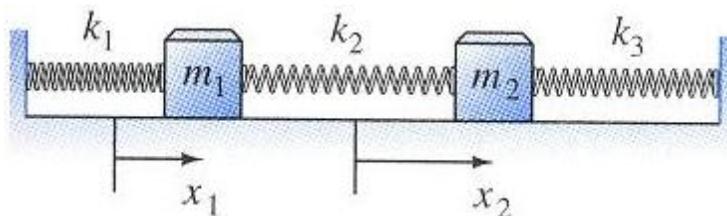
5. $k = 1, k_{12} = 3, m = 2$
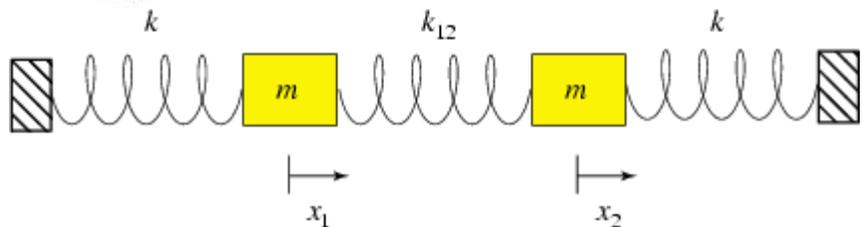
6. $m_1 = 1, m_2 = 2, m_3 = 3, k_1 = 1$,
   $k_2 = 4, c_1 = 2, c_2 = 5$
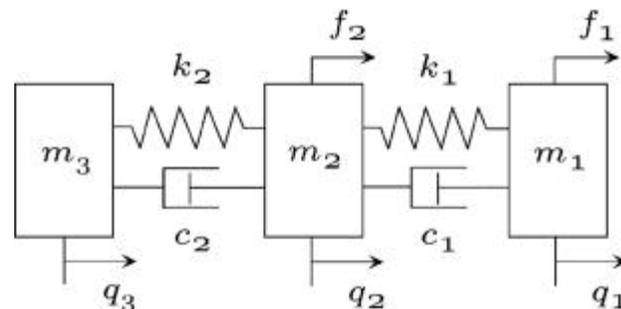
7. $m_1 = m_3 = 5, m_2 = m_4 = 1, k_1 = k_4 = k_5 = 3, k_2 = 1, k_3 = 10$
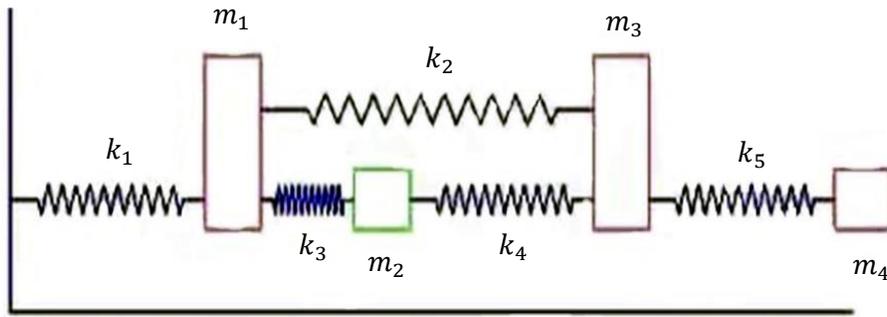
8. There are masses at dots marked 1, 2, 3 with $m = 1$. The springs are respectively: $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4$
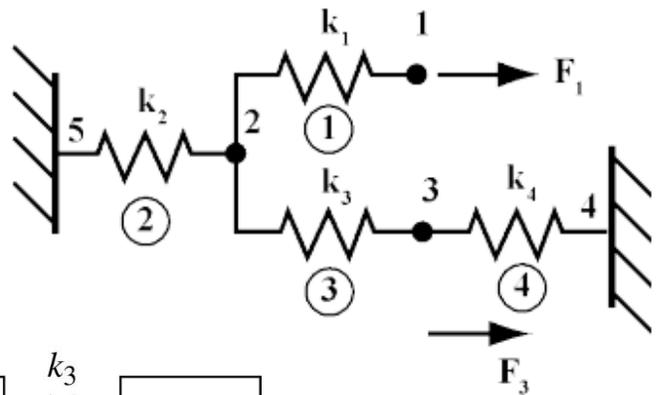
9. $m_1 = m_2 = m_3 = 1$,
    $k_1 = k_2 = k_3 = 1$
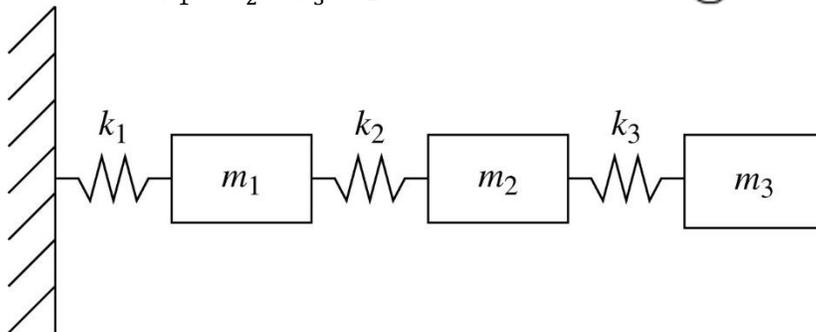
10. The masses are marked with arrows on the diagram (also indicating the x_1, x_2, x_3 locations.
    $m_1 = 1, m_2 = m_3 = 5, k_1 = k_2 = k_3 = 1$.
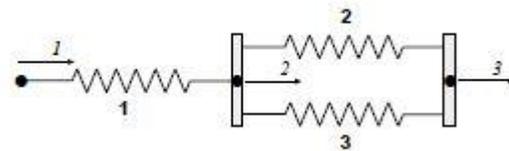
11. $m_1 = 10, m_2 = 15, k_1 = 4, k_2 = 1$,
    $c_2 = 3$
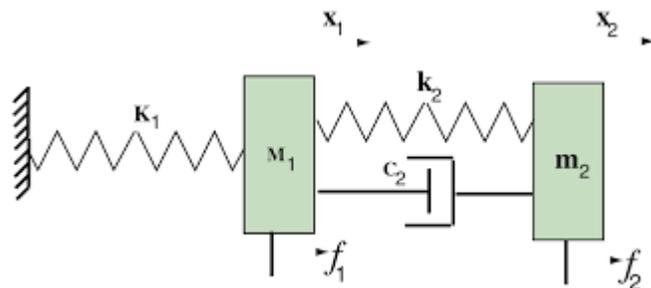
12. Values are noted on diagram.

$$k_1 = 4\text{N/m}$$

$$k_2 = 1\text{N/m} \qquad k_3 = 2\text{N/m}$$

u_1

u_2
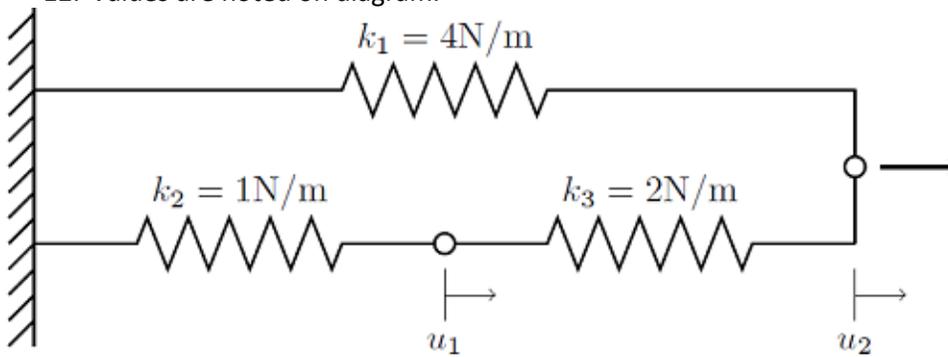
13. $m_1 = m_2 = m_3 = 10,$
$$k_{10} = k_{12} = k_{23} = k_{30} = 1,$$
$$k_{20} = 2, k_{13} = 5$$

k_{10}

k_{12}

m_1

k_{20}

X_1

k_{13}

m_2

k_{23}

X_2

m_3

k_{30}

X_3

14. $m_1 = m_2 = 20, k = 4, k_1 = k_2 = 16, c_1 = c_2 = 2$

k_1

k_2

k

m_1

m_2

c_1 f_1

f_2 c_2

x_1

x_2
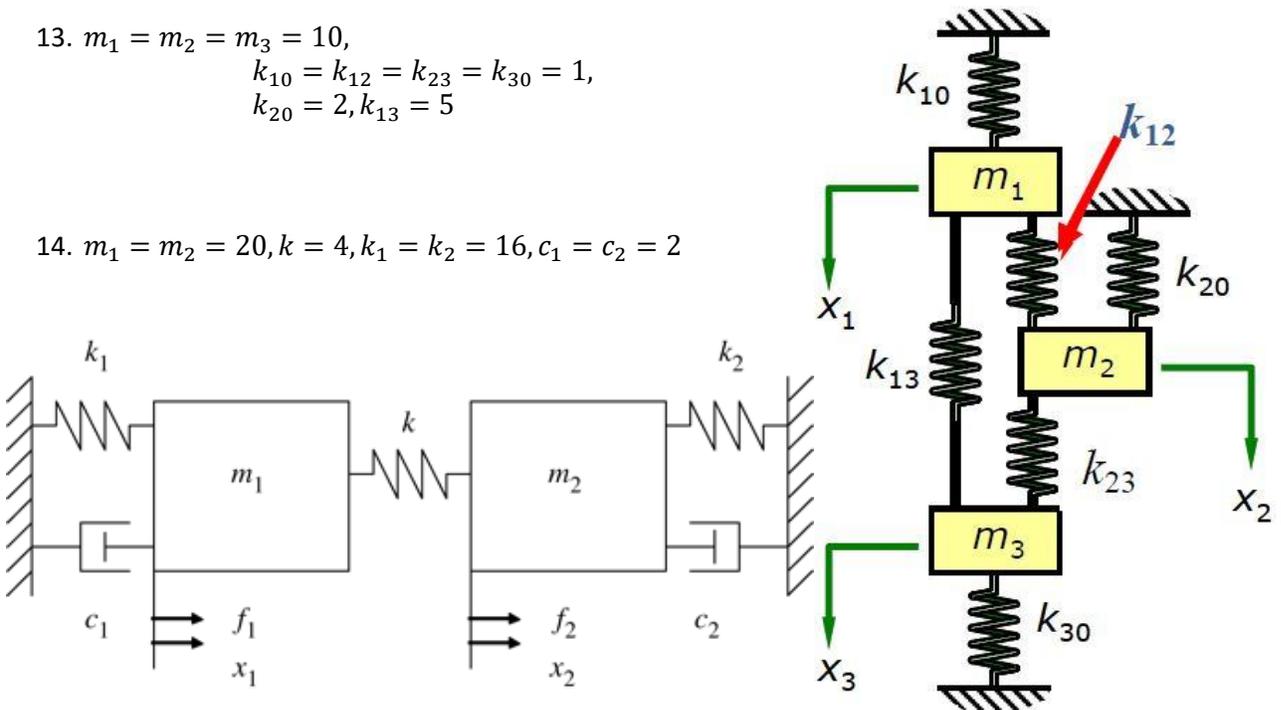
15. $m_1 = m_2 = m_3 = 60,$
$k_1 = k_3 = 5,\ k_{c1} =$
$k_{c2} = 6, k_2 = 12,$
$c_1 = c_3 = 3, c_2 = 10$

X_1

X_2

X_3

x

F_1

F_2

F_3

K_1

K_3

M_1 M_2 M_3

C_1

K_{C1}

K_{C2}

C_3

K_2

C_2

16. $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1, k_{1g} = k_{12} = k_{23} = k_{34} = k_{45} = k_{56} = 1$
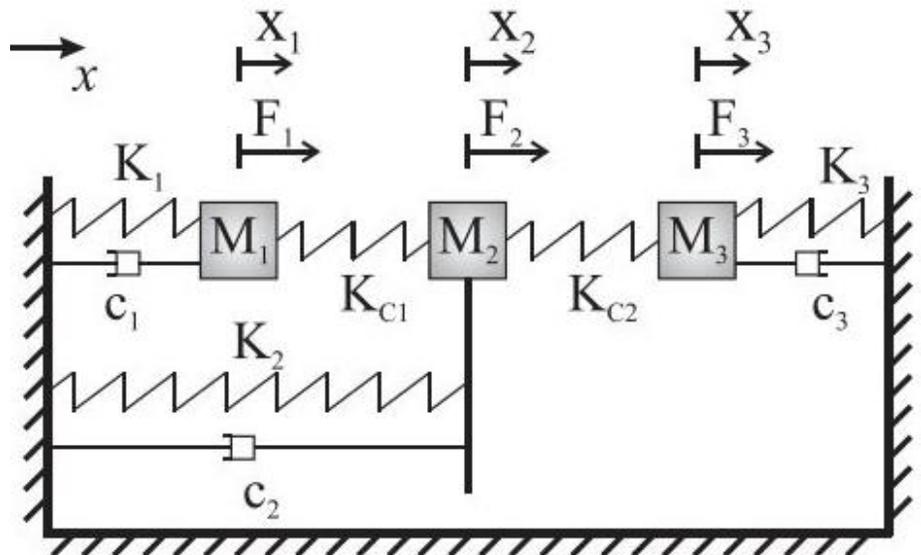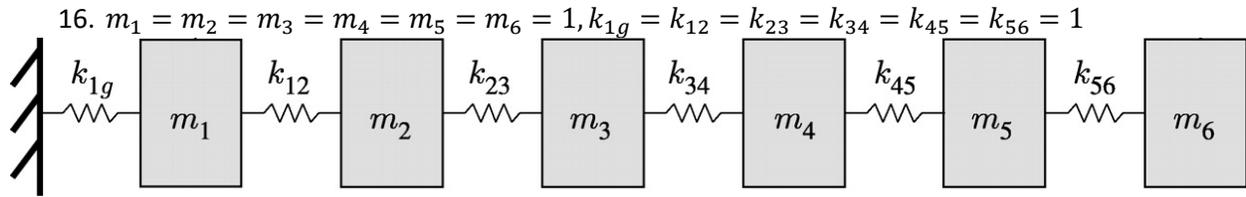
In addition to dealing with systems of equations in one direction of motion, if we want to model crystals or molecular bonds, we can increase the directions of motion to two or three dimensions.

Example 4. Consider the example below. Suppose that each mass is the same (call it $m = 1$ for simplicity). Each mass can move in either the x or y directions. Consider the mass in the second row, second column of the system. In the horizontal direction it has two springs exerting forces on it based on its own motion, and two nearby masses exerting forces on it. Resulting in the equation $m_{22}x_{22}'' = -k_{12h}x_{22} - k_{22h}x_{22} + k_{12h}x_{21} + k_{22h}x_{23}$ (the springs are also numbered "row by column" and whether they are horizontal or vertical). The forces acting on that mass vertically are $m_{22}y_{22}'' = -k_{12v}y_{22} - k_{22v}y_{22} + k_{12v}y_{12} + k_{22v}y_{32}$.
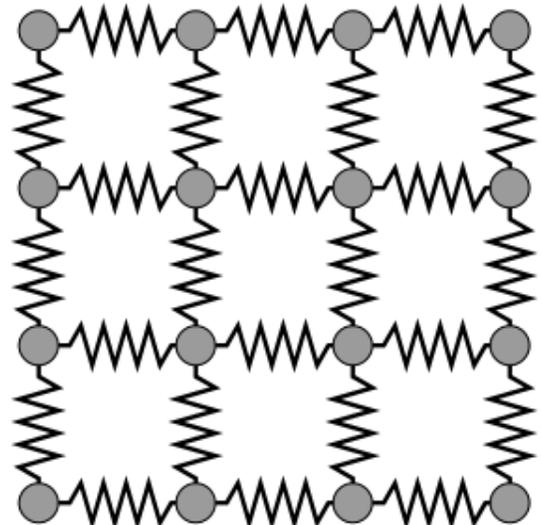
Each mass in the grid has a similar pair of equations.

Practice Problems.
1. For the same problem as in the example, set up a pair of differential equations to model one of the edge elements in the grid. Also Set up a pair of differential equation for the mass in the third row, third column of the grid. Assume for this problem that all the spring constants are 1.
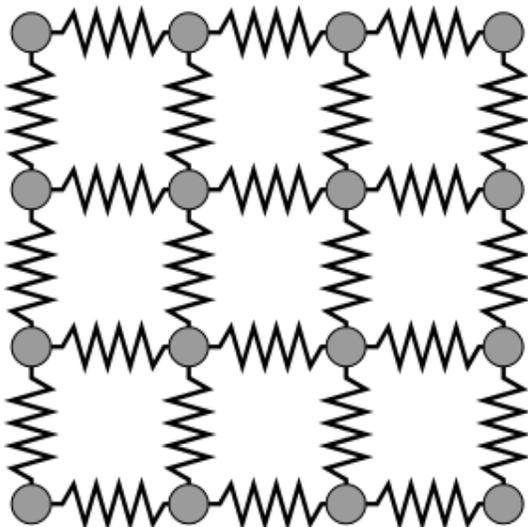
2. For the molecular "spring" model below, assume that the masses on the outside $m_1 = m_2 = m_3 = 6, m_4 = 1$ (m_4 is in the center). Also assume that the bonds form an isosceles triangle, with the angle between bonds being $60°$, and the angle to the central mass evenly divides the other angle. Use trigonometry to decompose the forces in the horizontal and vertical directions. Find the pair of differential equations for one of the outside masses.