

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Determine any singular points and classify them as regular or irregular.

a. $x^2(1+x)y'' + x(4-x^2)y' + (2+3x)y = 0$

$$y'' + \frac{x(4-x^2)}{x^2(1+x)} y' + \frac{(2+3x)}{x^2(1+x)} y = 0$$

$x=0, x=-1$ singular

$$\lim_{x \rightarrow 0} \frac{x(4-x^2)}{x^2(1+x)} \cdot x = 4 \quad \lim_{x \rightarrow 0} \frac{(2+3x)}{x^2(1+x)} x^2 = 2$$

$$\lim_{x \rightarrow -1} \frac{x(4-x^2)}{x^2(1+x)} = \frac{-3}{1} \quad \lim_{x \rightarrow -1} \frac{(2+3x)}{x^2(1+x)} (1+x)^2 = 0$$

both are regular

b. $x^4y'' + (x^2 \sin x)y' + (1 - \cos x)y = 0$

$$y'' + \frac{x^2 \sin x}{x^4} y' + \frac{1 - \cos x}{x^4} y = 0$$

$x=0$ singular

$$\lim_{x \rightarrow 0} \frac{x^2 \sin x}{x^4} \cdot x^2 = 1 \quad \lim_{x \rightarrow 0} \frac{(1 - \cos x) x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-\sin x}{2x} = -\frac{1}{2}$$

$x=0$ is regular

2. Determine if the vectors are linearly independent.

a. $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}$

b. $t, 2t+1, t^2-3t$

beat like vectors

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{one's coeff} \\ \leftarrow t \text{ coeff} \\ \leftarrow t^2 \text{ coeff} \end{array}$$

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ independent}$$

or use Wronskian

$$W = \begin{vmatrix} t & 2t+1 & t^2-3t \\ 1 & 2 & 2t-3 \\ 0 & 0 & 2 \end{vmatrix} =$$

independent.

$$2 \begin{vmatrix} t & 2t+1 \\ 1 & 2 \end{vmatrix} = 2t - (2t+1) = -1 \neq 0$$

$$\text{rref} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑
no pivot
not independent

also $\det A = 0$
so dependent