212 Homework #8 Key

1a. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ $\quad \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| = \lim_{x\to\infty} \left| \frac{\cancel{x^{2n}} x^2}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^2}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| = 0$

Converges on $(-\infty, \infty)$. radius of convergence is ∞

b. $\sum_{n=0}^{\infty} 2^n x^n$ $\quad \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{2^n x^n} \right| = \lim_{n\to\infty} |2x| < 1$

$\frac{-1 < 2x < 1}{2} \Rightarrow$ converges on $-\frac{1}{2} < x < \frac{1}{2}$ $\quad$ Radius of convergence $\frac{1}{2}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$ $\quad \lim_{n\to\infty} \left| \frac{(n+1)^2 (x+2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^2 (x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right| \cdot \left| \frac{x+2}{3} \right|$

$\lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right)^2 \right| \cdot \left| \frac{x+2}{3} \right| < 1$ $\quad -1 < \frac{x+2}{3} < 1 \Rightarrow \frac{-3 < x+2 < 3}{-2 \quad -2 \quad -2}$

radius of convergence is 3; converges on $-5 < x < 1$

d. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ $\quad \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \cdot |x-1| < 1$

$-1 < x-1 < 1$ $\quad$ radius of convergence is 1
$+1 \quad +1 \quad +1$
$0 < x < 2$ $\quad$ converges on $(0, 2)$

2a. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\qquad$ b. $\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n$

c. $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ $\qquad$ d. $x^3, x_0 = 1$

$f(x) = x^3 \quad f(1) = 1$

$f'(x) = 3x^2 \quad f'(1) = 3 \quad \frac{f'(1)}{1!} = 3$

$f''(x) = 6x \quad f''(1) = 6 \quad \frac{f''(1)}{2!} = \frac{6}{2} = 3$

$f'''(x) = 6 \quad f'''(1) = 6 \quad \frac{f'''(1)}{3!} = \frac{6}{6} = 1$

$f^{iv}(x) = 0 \quad 0$

$\vdots$

$x^3 = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$

e. $\frac{3x^2}{(1-x)^4}$ $\quad \frac{1}{1-x} = \sum_{n=0}^{\infty} ar^n$ $\quad (1-x)^{-1}$ $\quad \sum_{n=1}^{\infty} anr^{n-1} = (1-x)^{-2}$ $\quad \sum_{n=2}^{\infty} an(n-1)r^{n-2} = 2(1-x)^{-3}$

$\sum_{n=3}^{\infty} a\,n(n-1)(n-2)r^{n-3} = 6(1-x)^{-4} \Rightarrow \frac{1}{6}\sum_{n=0}^{\infty} a(n+3)(n+2)(n+1)r^n = \frac{a}{(1-x)^4}$

$\frac{1}{6} 3x^2 \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) x^n = \sum_{n=0}^{\infty} \frac{1}{2}(n+3)(n+2)(n+1) x^{n+2}$

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2f. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

g. $\frac{1}{1-x}, x_0 = -2 \quad = \frac{1}{3-(x+2)} \cdot \frac{1/3}{1/3} = \frac{1/3}{1-\frac{(x+2)}{3}}$

$a = 1/3,$

$r = \frac{x+2}{3}$

$1-x = 1-(x+2)+k$

$1-x-2+k$

$-1-x+k \Rightarrow k=2$

$3-(x+2) = 3-x-2 = 1-x$ ✓

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{x+2}{3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+1}(x+2)^n$$

3a. $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$

b. $\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}x^{n+1} = \sum_{n=1}^{\infty}(n+2)(n+1)n\,a_{n+2}x^n$

c. $\sum_{n=0}^{\infty} 2a_n x^{n+2} = \sum_{n=2}^{\infty} 2a_{n-2}x^n$

4a. $x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n =$

$\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n x^n + a_0 x^0 = \sum_{n=1}^{\infty}[n a_n + a_n]x^n + a_0$

$= \sum_{n=1}^{\infty} a_n(n+1)x^n + a_0$

b. $x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} + x^2\sum_{n=0}^{\infty} a_n x^n$

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+2} =$

$\sum_{n=1}^{\infty}(n+1)n\,a_{n+1}x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n =$

$\sum_{n=2}^{\infty}(n+1)n\,a_{n+1}x^n - \sum_{n=2}^{\infty} 2(n+1)a_{n+1}x^n - 2(1)a_1x^0 + \sum_{n=2}^{\infty} a_{n-2}x^n =$

$+ (2)(1)a_2x \qquad - 2(2)a_2x$

$\sum_{n=2}^{\infty}\left[(n+1)n\,a_{n+1} - 2(n+1)a_{n+1} + a_{n-2}x^n\right] - 2a_1 + (2a_2 - 4a_2)x$

$= -2a_2$

5. $y=\sum_{n=0}^{\infty} a_n x^n \qquad y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$y'''=\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3}$

$y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \qquad y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \qquad y'''=\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}x^n$

6. a. $\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}$ rref $\Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ yes. linearly independent

b. $\begin{bmatrix}1&3&2&4\\2&1&-1&3\\-2&0&1&-2\end{bmatrix}$ dependent more vectors than dimensions

c. $\begin{bmatrix}e^{-t}&e^{-t}&3e^{-t}\\2e^{-t}&e^{-t}&0\end{bmatrix}=e^{-t}\begin{bmatrix}1&1&3\\2&1&0\end{bmatrix}$ dependent, more vectors than dimensions

d. $\begin{bmatrix}1&-1&-2&-3\\2&0&-1&0\\2&3&1&-1\\3&1&0&3\end{bmatrix}$ rref $\Rightarrow \begin{bmatrix}1&0&0&2\\0&1&0&-3\\0&0&1&4\\0&0&0&0\end{bmatrix} \leftarrow$ no pivot

no pivot

not independent
not enough pivots

e. $\begin{bmatrix}2\sin t&\sin t\\\sin t&2\sin t\end{bmatrix}=\sin t\begin{bmatrix}2&1\\1&2\end{bmatrix} \Rightarrow 4-1=3$ determinant is non-zero so independent

7. a. $\vec{u}=\begin{bmatrix}-1\\2\end{bmatrix} \quad \vec{v}=\begin{bmatrix}4\\6\end{bmatrix}$ (real) $\vec{u}^T\vec{v}$

$\begin{bmatrix}-1&2\end{bmatrix}\begin{bmatrix}4\\6\end{bmatrix}=-4+12=8$

b. $\vec{u}=\begin{bmatrix}1-i\\2+3i\end{bmatrix} \quad \vec{v}=\begin{bmatrix}1-4i\\6i\end{bmatrix}$ complex $\vec{u}^T\overline{\vec{v}}$ ← conjugate

$\begin{bmatrix}1-i&2+3i\end{bmatrix}\begin{bmatrix}1+4i\\-6i\end{bmatrix}=(1-i)(1+4i)+(2+3i)(-6i)=1+4i-i+4-12i+18=$
$23-9i$

c. $\vec{u}=\begin{bmatrix}12\\3\\-5\end{bmatrix} \quad \vec{v}=\begin{bmatrix}2\\-3\\3\end{bmatrix}$ real

$\begin{bmatrix}12&3&-5\end{bmatrix}\begin{bmatrix}2\\-3\\3\end{bmatrix}=24-9-15=0$ (these vectors are orthogonal)

7d. $\vec{u} = \begin{bmatrix} 1+2i \\ 3i \\ -5i \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 1-3i \\ 3-i \end{bmatrix}$ complex

$$\begin{bmatrix} 1+2i & 3i & -5i \end{bmatrix} \begin{bmatrix} 2 \\ 1+3i \\ 3+i \end{bmatrix} = (1+2i)2 + 3i(1+3i) + (-5i)(3+i) =$$

$$2+4i+3i-9-15i+5 = -2-8i$$

e. $\vec{u} \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 & -5 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix} = -12+2+10+0 = 0$$ (these vectors are orthogonal

f. $\vec{u} = \begin{bmatrix} 3i \\ -2 \\ -5i \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1+4i \\ i \\ -2i \\ 6+i \end{bmatrix}$ complex

$$\begin{bmatrix} 3i & -2 & -5i & 0 \end{bmatrix} \begin{bmatrix} 1-4i \\ -i \\ 2i \\ 6-i \end{bmatrix} = 3i(1-4i) - 2(-i) - 5i(2i) + \cancel{0(6-i)}^{0} =$$

$$3i + 12 + 2i + 10 = 22 + 5i$$