212 Homework #13 key

1a. $m_1 x_1'' = -k_1x_1 - k_2x_2 + k_2x_2 \Rightarrow x_1'' = -5x_1 + 3x_2$

$m_2 x_2'' = -k_2x_2 + k_2x_1 \Rightarrow 5x_2'' = -3x_2 + 3x_1$

b. $m_1 x_1'' = -k_1x_1 - k_2x_1 + k_2x_2 \Rightarrow x_1'' = -5x_1 + 4x_2$

$m_1 x_2'' = -k_2x_2 - k_3x_2 + k_2x_1 \Rightarrow x_2'' = -5x_2 + 4x_1$

c. $m_2 x_2'' = -k_2x_2 + k_2x_1 \Rightarrow x_2'' = -x_2 + x_1$

$(m_1+m_2)x_1'' = -k_1x_1 - k_2x_1 + k_2x_2 \Rightarrow 2x_1'' = -5x_1 + x_2$

d. $m_1 x_1'' = -k_1x_1 - k_2x_1 + k_2x_2 \Rightarrow x_1'' = -2x_1 + x_2$

$m_2 x_2'' = -k_2x_2 - k_3x_2 + k_2x_1 + k_3x_3 \Rightarrow 2x_2'' = x_1 - 2x_2 + x_3$

$m_3 x_3'' = -k_3x_3 + k_3x_2 + p(t) \qquad 3x_3'' = x_2 - x_3 + p(t)$

2 a. $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\vec{x} \qquad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(5-\lambda)(1-\lambda)+3 = \lambda^2 - 6\lambda + 5 + 3 = 0$

$\lambda^2 - 6\lambda + 8 = 0 \quad (\lambda-4)(\lambda-2) = 0$

$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \quad \begin{matrix} 3x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_1 = 4, \ \lambda_2 = 2$

$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t} \Rightarrow c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 3 & -1 \end{array}\right] \quad \begin{matrix} c_1 = 7/2 \\ c_2 = -3/2 \end{matrix}$

$\vec{x}(t) = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t} - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t}$

b. $\vec{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix}\vec{x} \qquad (-\lambda)\begin{vmatrix} -\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 2 & -\lambda \\ -1 & 2 \end{vmatrix} =$

$(-\lambda)(-\lambda)(4-\lambda) - (4-\lambda) = 0 \qquad -(\lambda^3 - 4\lambda^2 - \lambda + 4) = 0$

$4\lambda^2 - \lambda^3 - 4 + \lambda = 0$

$\lambda^2(\lambda-4) - 1(\lambda-4) = 0$

$(\lambda^2-1)(\lambda-4) = 0$

$\lambda = \pm 1, \ \lambda = 4$

2b cont'd.

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \xRightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 = -x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \quad \lambda_1 = 1$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \xRightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = -2x_2 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \lambda_2 = -1$$

$$\begin{pmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{pmatrix} \xRightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 = -1/4\, x_3 \\ x_2 = -1/8\, x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix} \quad \lambda_3 = 4$$

$$\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix} e^{4t} \qquad \vec{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & -2 & 7 \\ -2 & -2 & -1 & 5 \\ 1 & 1 & 8 & 5 \end{array}\right] \Rightarrow \begin{matrix} c_1 = -6 \\ c_2 = 3 \\ c_3 = 1 \end{matrix}$$

$$\vec{x}(t) = \begin{pmatrix} 6 \\ 12 \\ -6 \end{pmatrix} e^{t} + \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix} e^{4t}$$

3a. $\vec{x}\,' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$

$$(3-\lambda)(-1-\lambda) + 8 = 0 \qquad \lambda = \frac{2 \pm \overset{4i}{\sqrt{4-20}}}{2}$$

$$\lambda^2 - 2\lambda - 3 + 8 = 0 \qquad = 1 \pm 2i$$

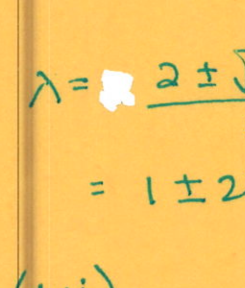
$$\lambda^2 - 2\lambda + 5 = 0$$

$$\begin{pmatrix} 3-1-2i & -2 \\ 4 & -1-1-2i \end{pmatrix} = \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \quad \begin{matrix} \cancel{2}x_1 = \frac{(1+i)x_2}{2} \\ x_2 = x_2 \end{matrix} \quad \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$e^{t} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} (\cos 2t + i \sin 2t) = e^{t} \begin{pmatrix} \cos 2t + i \sin 2t + i \cos 2t - \sin 2t \\ 2\cos 2t + 2i \sin 2t \end{pmatrix}$$

$$c_1 e^{t} \underbrace{\begin{pmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{pmatrix}}_{\text{Real part}} + c_2 e^{t} \underbrace{\begin{pmatrix} \sin 2t + \cos 2t \\ 2\sin 2t \end{pmatrix}}_{\text{Imaginary part}} = \vec{x}(t)$$

Use software to draw the direction field



Spirals outward

3b. $\vec{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}\vec{x}$

$(1-\lambda)(-1-\lambda) + 10 = 0$

$\lambda^2 - 1 + 10 = 0 \Rightarrow \lambda^2 + 9 = 0 \quad \lambda = \pm 3i$

$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}$

$\frac{-5x_1 = (1+3i)x_2}{5}$

$x_1 = \frac{(1+3i)x_2}{-5}$

$x_2 = x_2$

$\vec{v}_1 = \begin{pmatrix} 1+3i \\ -5 \end{pmatrix}$

$\begin{pmatrix} 1+3i \\ -5 \end{pmatrix}(\cos 3t + i \sin 3t) = \begin{pmatrix} \cos 3t + i\sin 3t + 3i\cos 3t - 3\sin 3t \\ -5\cos 3t - 5i\sin 3t \end{pmatrix}$

$\vec{x}(t) = c_1\begin{pmatrix} \cos 3t - 3\sin 3t \\ -5\cos 3t \end{pmatrix} + c_2\begin{pmatrix} \sin 3t + 3\cos 3t \\ -5\sin 3t \end{pmatrix}$

4a. $\vec{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}\vec{x} \Rightarrow (\alpha-\lambda)(\alpha-\lambda) + 1 = 0$

$\lambda^2 - 2\alpha\lambda + (\alpha^2+1) = 0$

$\lambda = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2+1)}}{2}$

$= \frac{2\alpha \pm \sqrt{4\alpha^2 - 4\alpha^2 - 4}}{2} = \frac{2\alpha \pm 2i}{2}$

$= \alpha \pm i$

behavoir changes at $\alpha = 0$

when $\alpha < 0$ graph sperals in toward origin

when $\alpha = 0$ graph has a stable orbit

when $\alpha > 0$ graph speral outward away from origin

b. $\vec{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix}\vec{x}$

$(2-\lambda)(-2-\lambda) + 5\alpha = 0$

$\lambda^2 - 4 + 5\alpha = 0$

$\lambda = \pm\sqrt{4-5\alpha}$

$4 - 5\alpha = 0$

$4 = 5\alpha$

$\alpha = 4/5$

behavoir changes at $\alpha = 5/4$

when $\alpha = 4/5$ one root is constant, other is a multiple of $t \to \infty$ w/ time

when $\alpha < 4/5$ there are two distinct real roots w/ opposite signs that produce a saddle point

when $\alpha > 4/5$ solutions are pure imaginary and a stable orbit results.

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5. $m_1 \dfrac{d^2x_1}{dt^2} = -(k_1+k_2)x_1 + k_2x_2$

$m_2 \dfrac{d^2x_2}{dt^2} = k_2x_1 - (k_2+k_3)x_2$ $\qquad m_1 = m_2 = k_1 = k_2 = k_3 = 1$

$x_1'' = -2x_1 + x_2$

$x_2'' = x_1 - 2x_2$

a. convert to first order system: $x_3 = x_1'$ $\quad x_4 = x_2'$

$x_3' = x_1''$ $\quad x_4' = x_2''$

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_4 \\ x_3' &= -2x_1 + x_2 \\ x_4' &= x_1 - 2x_2 \end{aligned}$$

$$\vec{x}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \vec{x}$$

b. coff matrix A.

c. $\lambda_{1,2} = \pm\sqrt{3}\, i \quad \lambda_{3,4} = \pm i$

$\vec{v}_{1,2} = \begin{pmatrix} \sqrt{3}i \\ -\sqrt{3}i \\ -3 \\ 3 \end{pmatrix} \quad \vec{v}_{3,4} = \begin{pmatrix} i \\ i \\ 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} \sqrt{3}i \\ -\sqrt{3}i \\ -3 \\ 3 \end{pmatrix}(\cos\sqrt{3}t + i\sin\sqrt{3}t) = \begin{pmatrix} \sqrt{3}i\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ -\sqrt{3}i\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t \\ -3\cos\sqrt{3}t - 3i\sin\sqrt{3}t \\ 3\cos\sqrt{3}t + i3\sin\sqrt{3}t \end{pmatrix}$$

$$\begin{pmatrix} i \\ i \\ 1 \\ 1 \end{pmatrix}(\cos t + i\sin t) = \begin{pmatrix} i\cos t - \sin t \\ i\cos t - \sin t \\ \cos t + i\sin t \\ \cos t + i\sin t \end{pmatrix}$$

$$\vec{x}(t) = c_1\begin{pmatrix} -\sqrt{3}\sin\sqrt{3}t \\ \sqrt{3}\sin\sqrt{3}t \\ -3\cos\sqrt{3}t \\ 3\cos\sqrt{3}t \end{pmatrix} + c_2\begin{pmatrix} \sqrt{3}\cos\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t \\ -3\sin\sqrt{3}t \\ 3\sin\sqrt{3}t \end{pmatrix} + c_3\begin{pmatrix} -\sin t \\ -\sin t \\ \cos t \\ \cos t \end{pmatrix} + c_4\begin{pmatrix} \cos t \\ \cos t \\ \sin t \\ \sin t \end{pmatrix}$$

the fundamental modes of vibration have frequency 1 and $\sqrt{3}$

5e. $c_1\begin{pmatrix}0\\0\\-3\\3\end{pmatrix} + c_2\begin{pmatrix}\sqrt{3}\\-\sqrt{3}\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + c_4\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\3\\0\\0\end{pmatrix}$

$\left[\begin{array}{cccc|c} 0 & \sqrt{3} & 0 & 1 & -1\\ 0 & -\sqrt{3} & 0 & 1 & 3\\ -3 & 0 & 1 & 0 & 0\\ 3 & 0 & 1 & 0 & 0 \end{array}\right]$ $\quad c_1 = 0,\ c_2 = \frac{\sqrt{3}}{2}$, $c_3 = 0\quad c_4 = 1$

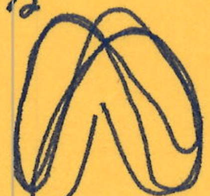
$\vec{x}(t) = \begin{pmatrix}-\frac{3}{2}\cos\sqrt{3}t\\ \frac{3}{2}\cos\sqrt{3}t\\ \frac{3\sqrt{3}}{2}\sin\sqrt{3}t\\ \frac{3\sqrt{3}}{2}\sin\sqrt{3}t\end{pmatrix} + \begin{pmatrix}\cos t\\ \cos t\\ \sin t\\ \sin t\end{pmatrix}$

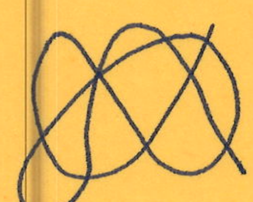
original variables

derivatives

yes, the solution is periodic separately (but not together)

$x_3 x_4$





$x_1 x_2$

f. answers will vary.

6a. $\vec{x}' = \begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix}\vec{x}$

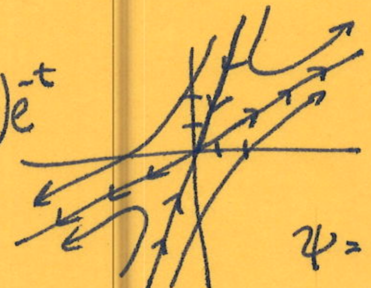
$(3-\lambda)(-2-\lambda)+4=0$
$\lambda^2-\lambda-6+4=0$
$\lambda^2-\lambda-2=0$
$(\lambda-2)(\lambda+1)=0$

$\lambda_1 = 2,\ \lambda_2 = -1$

$\begin{bmatrix}1 & -2\\ 2 & -4\end{bmatrix}$ $\quad x_1 = 2x_2$, $x_2 = x_2$, $\vec{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$

$\begin{bmatrix}4 & -2\\ 2 & -1\end{bmatrix}$ $\quad 2x_1 = x_2$, $x_2 = x_2$, $\vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$

$\vec{x}(t) = c_1\begin{pmatrix}2\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e^{-t}$



$\vec{x}$ increas exponentially for most initial conditions, approaching the vector $\begin{bmatrix}2\\1\end{bmatrix}$ as $t\to\infty$

$\Psi = \begin{pmatrix}2e^{2t} & e^{-t}\\ e^{2t} & 2e^{-t}\end{pmatrix}$

b. $\vec{x}' = \begin{pmatrix}2 & 2+i\\ -1 & -1-i\end{pmatrix}\vec{x}$

$(2-\lambda)(-1-i-\lambda)+2+i=0$
$-2-2i-2\lambda+\lambda+i\lambda+\lambda^2+2+i=0$
$\lambda^2+\lambda(-1+i)-i=0$
$\lambda = \dfrac{-(-1+i)\pm\sqrt{(-1+i)^2-4(-i)}}{2} = \dfrac{1-i\pm\sqrt{-2i+4i}}{2}$

$(-1+i)(-1+i) = 1-i-i-1$
$= -2i$

$\sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4}, e^{i5\pi/4}$
$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i \quad -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

$\dfrac{1-i\pm\sqrt{2i}}{2} = \dfrac{1-i\pm\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\right)}{2}$

$\dfrac{1-\cancel{i}+(1+\cancel{i})}{2} = 1$

$\dfrac{1-i-\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\right)}{2} = \dfrac{1-i-1-i}{2} = -i$

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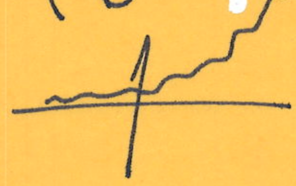
6b cont'd $\lambda_1 = 1, \lambda_2 = i$

$$\begin{pmatrix} 2-1 & 2+i \\ -1 & -1-i-1 \end{pmatrix} = \begin{pmatrix} 1 & 2+i \\ -1 & -2-i \end{pmatrix} \quad \begin{matrix} x_1 = -(2+i)x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{pmatrix} 2+i \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2+i & 2+i \\ -1 & -1-i+i \end{pmatrix} = \begin{pmatrix} 2+i & 2+i \\ -1 & -1 \end{pmatrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 2+i \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-it} \qquad \Psi = \begin{pmatrix} (2+i)e^{t} & -e^{-it} \\ -e^{t} & e^{-it} \end{pmatrix}$$



real part $c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cos t$

we can write this solution in complex #'s since the original matrix is complex – but graphing trajectories is problematic

c. $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}$ $\quad (-2-\lambda)(-2-\lambda) - 1 = 0$ $\quad \lambda^2 + 4\lambda + 4 - 1 = 0$ $\quad \lambda^2 + 4\lambda + 3 = 0$ $\quad (\lambda+3)(\lambda+1) = 0$ $\quad \lambda_1 = -3, \lambda_2 = -1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

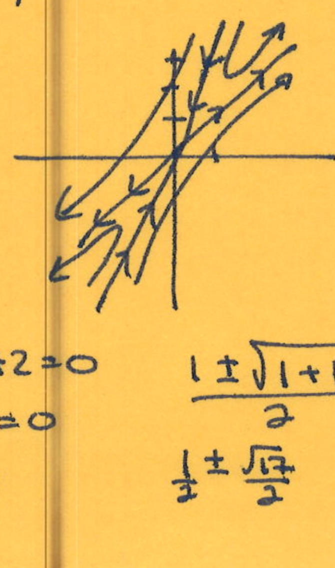
$$\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \qquad \Psi = \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}$$

d. $t\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}$ $\quad (2-\lambda)(-2-\lambda) + 3 = 0$ $\quad \lambda^2 - 4 + 3 = 0$ $\quad \lambda^2 - 1 = 0$ $\quad (\lambda-1)(\lambda+1) = 0$ $\quad \lambda = \pm 1$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \quad \begin{matrix} 3x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-1}$$

$$\Psi = \begin{pmatrix} t & 1/t \\ t & 3/t \end{pmatrix}$$



7a. $\vec{x}' = \begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{t} \\ t \end{pmatrix}$ $\quad (3-\lambda)(-2-\lambda) + 2 = 0$ $\quad \lambda^2 - \lambda - 6 + 2 = 0$ $\quad \lambda^2 - \lambda - 4 = 0$ $\quad \dfrac{1 \pm \sqrt{1+16}}{2}$ $\quad \dfrac{1}{2} \pm \dfrac{\sqrt{17}}{2}$

7a cont'd

$$\begin{pmatrix} 3-\frac{1}{2}-\frac{\sqrt{17}}{2} & -1 \\ 2 & -2-\frac{1}{2}-\frac{\sqrt{17}}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2}-\frac{\sqrt{17}}{2} & -1 \\ 2 & -\frac{5}{2}-\frac{\sqrt{17}}{2} \end{pmatrix}$$

$2x_1 = \left(\frac{5}{2}+\frac{\sqrt{17}}{2}\right)x_2$

$x_1 = \left(\frac{5}{4}+\frac{\sqrt{17}}{4}\right)x_2$ $\quad \vec{V}_1 = \begin{pmatrix} 5+\sqrt{17} \\ 4 \end{pmatrix}$

$x_2 = \quad x_2$

$\vec{V}_2 = \begin{pmatrix} 5-\sqrt{17} \\ 4 \end{pmatrix}$

$$\vec{X}(t) = c_1 \begin{pmatrix} 5+\sqrt{17} \\ 4 \end{pmatrix} e^{(1/2+\sqrt{17}/2)t} + c_2 \begin{pmatrix} 5-\sqrt{17} \\ 4 \end{pmatrix} e^{\left(1/2-\frac{\sqrt{17}}{2}\right)t}$$

$$\Psi = \begin{pmatrix} (5+\sqrt{17})e^{(1/2+\frac{\sqrt{17}}{2})t} & (5-\sqrt{17})e^{(1/2-\sqrt{17}/2)t} \\ 4e^{(1/2+\frac{\sqrt{17}}{2})t} & 4e^{(1/2-\sqrt{17}/2)t} \end{pmatrix}$$

$$\begin{pmatrix} e^t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$$

$$X_p(t) = \begin{pmatrix} a \\ b \end{pmatrix} e^t + \begin{pmatrix} c \\ d \end{pmatrix} t + \begin{pmatrix} e \\ f \end{pmatrix} 1 \qquad X' = \begin{pmatrix} a \\ b \end{pmatrix} e^t + \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} e^t = \begin{pmatrix} 3a-2b \\ 2a-2b \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t = \begin{pmatrix} a \\ b \end{pmatrix} e^t$$

$$\begin{pmatrix} 2a-2b \\ 2a-3b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \left[\begin{array}{cc|c} 2 & -2 & 1 \\ 2 & -3 & 0 \end{array}\right]$$

$a = 3/2$
$b = 1$

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix}\left[\begin{pmatrix} c \\ d \end{pmatrix} t + \begin{pmatrix} e \\ f \end{pmatrix}\right] = \left[\begin{pmatrix} 3c-d \\ 2c-2d \end{pmatrix} t + \begin{pmatrix} 3e-f \\ 2e-2f \end{pmatrix}\right] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 3 & -1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 \\ -1 & 0 & 3 & -1 & 0 \\ 0 & -1 & 2 & -2 & 0 \end{array}\right]$$

$c = -\frac{1}{4}$
$d = -3/4$
$e = \frac{1}{16}$
$f = \frac{7}{16}$

$$\vec{X}_p(t) = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -1/4 \\ -3/4 \end{pmatrix} t + \begin{pmatrix} 1/16 \\ 7/16 \end{pmatrix} \qquad \vec{X}(t) = \Psi \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \vec{X}_p$$

(this solution used the method of undetermined coefficients)

7b. $\vec{X}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{X} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$

$(1-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 - 2\lambda + 1 - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda-3)(\lambda+1) = 0$

$\lambda_1 = 3, \ \lambda_2 = -1$

$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$ $\quad 2x_1 = x_2$, $\ x_2 = x_2$

$\vec{V}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

7b cont'd

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \quad \begin{matrix} 2x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Psi = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

$$\vec{x}_p = \Psi \int \Psi^{-1} g \, dt \qquad \Psi^{-1} = \frac{1}{\underbrace{-2e^{2t} - 2e^{2t}}_{-4e^{2t}}} \begin{pmatrix} -2e^{-t} & -e^{-t} \\ -2e^{3t} & e^{3t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-3t} & \frac{1}{4}e^{-3t} \\ \frac{1}{2}e^{t} & -\frac{1}{4}e^{t} \end{pmatrix}$$

$$\Psi^{-1} g = \begin{pmatrix} \frac{1}{2}e^{-3t} & \frac{1}{4}e^{-3t} \\ \frac{1}{2}e^{t} & -\frac{1}{4}e^{t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} e^{-2t} - \frac{1}{4}e^{-2t} \\ e^{2t} + \frac{1}{4}e^{2t} \end{pmatrix} = \begin{pmatrix} \frac{3}{4}e^{-2t} \\ \frac{5}{4}e^{2t} \end{pmatrix}$$

$$\int \begin{pmatrix} \frac{3}{4}e^{-2t} \\ \frac{5}{4}e^{2t} \end{pmatrix} dt = \begin{pmatrix} -\frac{3}{8}e^{-2t} \\ \frac{5}{8}e^{2t} \end{pmatrix} \qquad \Psi \int \Psi^{-1} g \, dt = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} -\frac{3}{8}e^{-2t} \\ \frac{5}{8}e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{3}{8}e^t + \frac{5}{8}e^t \\ -\frac{3}{4}e^t - \frac{5}{4}e^t \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix} e^t = \vec{x}_p$$

this solution used variation of parameters

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix} e^t$$

7c. $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \quad \begin{matrix} x_1 = (2+i)x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$(2-\lambda)(-2-\lambda) + 5 = 0$$
$$\lambda^2 - 4 + 5 = 0$$
$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

$$\vec{v}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 2\cos t + 2i\sin t + i\cos t - \sin t \\ \cos t + i \sin t \end{pmatrix}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \qquad \Psi = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}$$

$$\Psi^{-1} = \frac{1}{-1} \begin{pmatrix} \sin t & -2\sin t - \cos t \\ -\cos t & 2\cos t - \sin t \end{pmatrix} \qquad \Psi^{-1} g = \begin{pmatrix} -\sin t & 2\sin t + \cos t \\ \cos t & \sin t - 2\cos t \end{pmatrix} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$|\Psi| = \cancel{2\cos t \sin t} - \sin^2 t - \cancel{2\cos t \sin t} - \cos^2 t = -1$$

$$= \begin{pmatrix} \sin t \cos t + 2\sin^2 t + \cos t \sin t \\ -\cos^2 t + \sin^2 t - 2\cos t \sin t \end{pmatrix}$$

7c. cont'd

$$\Psi^{-1} g = \begin{pmatrix} 2\sin t\cos t + 2\sin^2 t \\ \sin^2 t - \cos^2 t - 2\sin t\cos t \end{pmatrix} = \begin{pmatrix} \sin 2t + 1 - \cos 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

$$\int \Psi^{-1} g \, dt = \int \begin{pmatrix} \sin 2t + 1 - \cos 2t \\ -\cos 2t - \sin 2t \end{pmatrix} dt = \begin{pmatrix} -\frac{1}{2}\cos 2t + t - \frac{1}{2}\sin 2t \\ -\frac{1}{2}\sin 2t + \frac{1}{2}\cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t + t - \sin t\cos t \\ -\sin t\cos t + \frac{1}{2}\cos^2 t - \frac{1}{2}\sin^2 t \end{pmatrix}$$

$$\Psi \int \Psi^{-1} g = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t + t - \sin t\cos t \\ -\sin t\cos t + \frac{1}{2}\cos^2 t - \frac{1}{2}\sin^2 t \end{pmatrix}$$

top entry.

$$-\cos^3 t + \cancel{\cos t\sin^2 t} + 2t\cos t - 2\cos^2 t\sin t + \frac{1}{2}\sin t\cos^2 t - \frac{1}{2}\sin^3 t - t\sin t$$

$$+ \cancel{\sin^2 t\cos t} - 2\sin^2 t\cos t + \cancel{\sin t\cos^2 t} - \sin^3 t - \cancel{\sin t\cos^2 t} + \frac{1}{2}\cos^3 t - \frac{1}{2}\sin^2 t\cos t$$

$$-\frac{1}{2}\cos^3 t - \frac{3}{2}\sin^3 t + 2t\cos t - t\sin t - \frac{1}{2}\sin^2 t\cos t - \frac{3}{2}\sin t\cos^2 t$$

$$-\frac{3}{2}\sin t(\cancel{\sin^2 t + \cos^2 t}) - \frac{1}{2}\cos t(\cancel{\cos^2 t + \sin^2 t}) + 2t\cos t - t\sin t$$

$$\sin t(-t - \tfrac{3}{2}) + \cos t(2t - \tfrac{1}{2})$$

bottom entry

$$-\frac{1}{2}\cos^3 t + \frac{1}{2}\sin^2 t\cos t + t\cos t - \sin t\cos^2 t - \sin^2 t\cos t + \frac{1}{2}\cos^2 t\sin t - \frac{1}{2}\sin^3 t$$

$$t\cos t - \frac{1}{2}\sin^2 t\cos t - \frac{1}{2}\cos^2 t\sin t - \frac{1}{2}\cos^3 t - \frac{1}{2}\sin^3 t$$

$$t\cos t - \frac{1}{2}\cos t(\cancel{\sin^2 t + \cos^2 t}) - \frac{1}{2}\sin t(\cancel{\cos^2 t + \sin^2 t})$$

$$= t\cos t - \frac{1}{2}\cos t - \frac{1}{2}\sin t = \cos t(t - \tfrac{1}{2}) - \tfrac{1}{2}\sin t$$

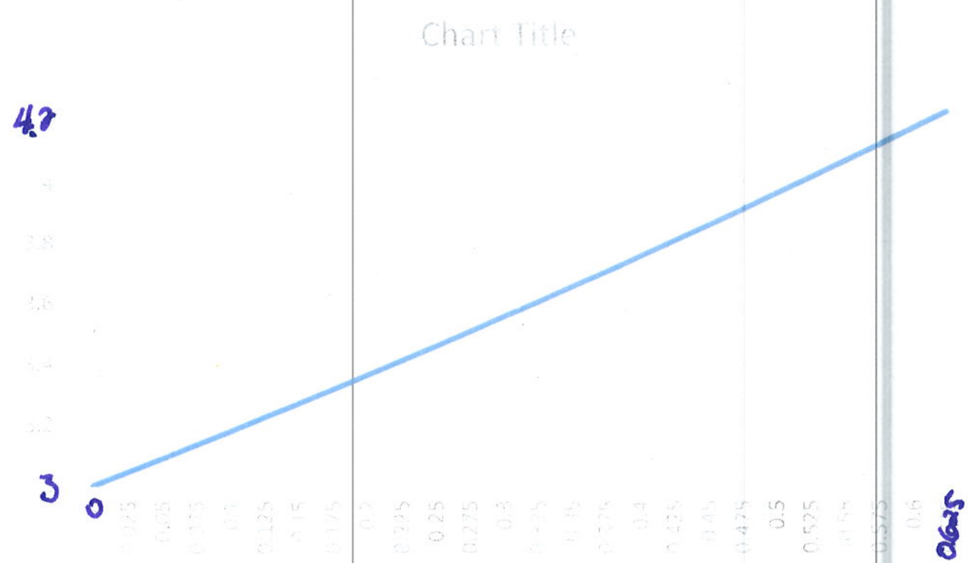
$$\vec{x}_p = \begin{pmatrix} \cos t(2t - \frac{1}{2}) - \sin(t + \frac{3}{2}) \\ \cos t(t - \frac{1}{2}) - \frac{1}{2}\sin t \end{pmatrix}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} + \begin{pmatrix} (2t - \frac{1}{2})\cos t - (t + \frac{3}{2})\sin t \\ (t - \frac{1}{2})\cos t - \frac{1}{2}\sin t \end{pmatrix}$$

8 → attached on Excel sheets

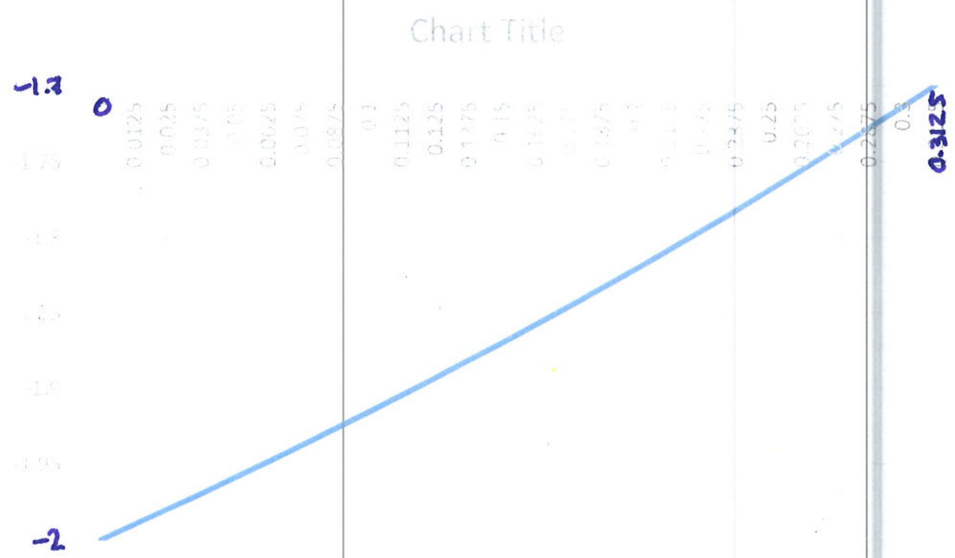
8a

Step (n)	t_n	y_n	m_n=f(t_n,y_n)	Delta_t=h	y_(n+1)
0	0	3	1.732050808	0.025	3.043301
1	0.025	3.043301	1.751656722	0.025	3.087093
2	0.05	3.087093	1.771183979	0.025	3.131372
3	0.075	3.131372	1.790634605	0.025	3.176138
4	0.1	3.176138	1.810010539	0.025	3.221388
5	0.125	3.221388	1.829313646	0.025	3.267121
6	0.15	3.267121	1.848545714	0.025	3.313335
7	0.175	3.313335	1.867708462	0.025	3.360028
8	0.2	3.360028	1.886803544	0.025	3.407198
9	0.225	3.407198	1.905832548	0.025	3.454844
10	0.25	3.454844	1.924797006	0.025	3.502963
11	0.275	3.502963	1.943698392	0.025	3.551556
12	0.3	3.551556	1.962538127	0.025	3.600619
13	0.325	3.600619	1.98131758	0.025	3.650152
14	0.35	3.650152	2.000038073	0.025	3.700153
15	0.375	3.700153	2.01870088	0.025	3.750621
16	0.4	3.750621	2.037307234	0.025	3.801553
17	0.425	3.801553	2.055858324	0.025	3.85295
18	0.45	3.85295	2.074355299	0.025	3.904809
19	0.475	3.904809	2.092799271	0.025	3.957129
20	0.5	3.957129	2.111191315	0.025	4.009909
21	0.525	4.009909	2.129532473	0.025	4.063147
22	0.55	4.063147	2.147823751	0.025	4.116842
23	0.575	4.116842	2.166066125	0.025	4.170994
24	0.6	4.170994	2.184260541	0.025	4.225601
25	0.625	4.225601	2.202407915	0.025	4.280661



8b

Step (n)	t_n	y_n	m_n=f(t_n,y_n)	Delta_t=h	y_(n+1)
0	0	-2	0.8	0.0125	-1.99
1	0.0125	-1.99	0.811450374	0.0125	-1.97986
2	0.025	-1.97986	0.823096279	0.0125	-1.96957
3	0.0375	-1.96957	0.83494422	0.0125	-1.95913
4	0.05	-1.95913	0.847000994	0.0125	-1.94854
5	0.0625	-1.94854	0.859273704	0.0125	-1.9378
6	0.075	-1.9378	0.871769781	0.0125	-1.92691
7	0.0875	-1.92691	0.884496995	0.0125	-1.91585
8	0.1	-1.91585	0.897463484	0.0125	-1.90463
9	0.1125	-1.90463	0.91067777	0.0125	-1.89325
10	0.125	-1.89325	0.924148781	0.0125	-1.8817
11	0.1375	-1.8817	0.937885882	0.0125	-1.86997
12	0.15	-1.86997	0.951898892	0.0125	-1.85807
13	0.1625	-1.85807	0.966198122	0.0125	-1.846
14	0.175	-1.846	0.980794396	0.0125	-1.83374
15	0.1875	-1.83374	0.99569909	0.0125	-1.82129
16	0.2	-1.82129	1.010924165	0.0125	-1.80865
17	0.2125	-1.80865	1.026482203	0.0125	-1.79582
18	0.225	-1.79582	1.04238645	0.0125	-1.78279
19	0.2375	-1.78279	1.058650855	0.0125	-1.76956
20	0.25	-1.76956	1.075290123	0.0125	-1.75612
21	0.2625	-1.75612	1.092319759	0.0125	-1.74246
22	0.275	-1.74246	1.109756126	0.0125	-1.72859
23	0.2875	-1.72859	1.127616501	0.0125	-1.7145
24	0.3	-1.7145	1.145919142	0.0125	-1.70017
25	0.3125	-1.70017	1.164683351	0.0125	-1.68561



8b

Step (n)	t_n	y_n	m_n=f(t_n,y_n)	Delta_t=h	y_(n+1)
0	0	1	2.	0.0500	1.1
1	0.05	1.1	2.05	0.05	1.2025
2	0.1	1.2025	2.105	0.05	1.30775
3	0.15	1.30775	2.1655	0.05	1.416025
4	0.2	1.416025	2.23205	0.05	1.527628
5	0.25	1.527628	2.305255	0.05	1.64289
6	0.3	1.64289	2.3857805	0.05	1.762179
7	0.35	1.762179	2.47435855	0.05	1.885897
8	0.4	1.885897	2.571794405	0.05	2.014487
9	0.45	2.014487	2.678973846	0.05	2.148436
10	0.5	2.148436	2.79687123	0.05	2.288279
11	0.55	2.288279	2.926558353	0.05	2.434607
12	0.6	2.434607	3.069214188	0.05	2.588068
13	0.65	2.588068	3.226135607	0.05	2.749375
14	0.7	2.749375	3.398749168	0.05	2.919312
15	0.75	2.919312	3.588624085	0.05	3.098743
16	0.8	3.098743	3.797486493	0.05	3.288618
17	0.85	3.288618	4.027235142	0.05	3.489979
18	0.9	3.489979	4.279958657	0.05	3.703977
19	0.95	3.703977	4.557954522	0.05	3.931875
20	1	3.931875	4.863749975	0.05	4.175062
21	1.05	4.175062	5.200124972	0.05	4.435069
22	1.1	4.435069	5.570137469	0.05	4.713576
23	1.15	4.713576	5.977151216	0.05	5.012433
24	1.2	5.012433	6.424866338	0.05	5.333676
25	1.25	5.333676	6.917352972	0.05	5.679544