212 Homework #12 key

1a. $u'' + \frac{1}{2}u' + 2u = 0$ let $u = x_1$ $u' = x_1' = x_2$ $x_2' = u''$

$x_2' + 1/2\, x_2 + 2x_1 = 0 \Rightarrow x_1' = x_2$

$x_2' = -2x_1 - 1/2\, x_2$

$$[\vec{x}]' = \begin{bmatrix} 0 & 1 \\ -2 & -1/2 \end{bmatrix}\vec{x}$$

b. $u^{iv} - u = 0$ $u = x_1$ $u' = x_1' = x_2$ $u'' = x_2' = x_3$ $u''' = x_3' = x_4$ $u^{iv} = x_4'$

$x_4' - x_1 = 0$

$x_1' = x_2$
$x_2' = x_3$
$x_3' = x_4$
$x_4' = x_1$

$$\Rightarrow \vec{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\vec{x}$$

c. $t^2u'' + tu' + (t^2 - \frac{1}{4})u = 0$ $u = x_1$ $u' = x_1' = x_2$ $x_2' = u''$

$t^2 x_2' + t x_2 + (t^2 - \frac{1}{4})x_1 = 0$

$x_2' = -\frac{1}{t}x_2 - (1 - \frac{1}{4t^2})x_1$

$x_1' = x_2$

$x_2' = (\frac{1}{4t^2} - 1)x_1 - \frac{1}{t}x_2$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ \frac{1}{4t^2} - 1 & -\frac{1}{t} \end{bmatrix}\vec{x}$$

d. $u'' + \frac{1}{4}u' + 4u = 2\cos 3t$ $u(0) = 1$, $u'(0) = -2$

$u = x_1$, $x_1' = x_2 = u'$ $x_2' = u''$

$x_1' = x_2$

$x_2' = -\frac{1}{4}x_2 - 4x_1 + 2\cos 3t$

$$\vec{x} = \begin{bmatrix} 0 & 1 \\ -4 & -1/4 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ 2\cos 3t \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2a. $x_1' = 3x_1 - 2x_2$

$x_2' = 2x_1 - 2x_2$

$\frac{x_1' - 3x_1}{-2} = x_2 = -\frac{1}{2}x_1' + \frac{3}{2}x_1$

$x_2' = -\frac{1}{2}x_1'' + \frac{3}{2}x_1'$

$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$
$r = 2, r = -1$

$-\frac{1}{2}x_1'' + \frac{3}{2}x_1' = 2x_1 - 2(-1/2\, x_1' + \frac{3}{2}x_1)$

$-2(-\frac{1}{2}x_1'' + \frac{3}{2}x_1' = 2x_1 + x_1' - 3x_1)$

$x_1'' - 3x_1' = -4x_1 - 2x_1' + 6x_1$

$x_1'' - x_1' - 2x_1 = 0 \Rightarrow u'' - u' - 2u = 0$

$u(0) = 3$

$u'(0) = 3(3) - 2(1/2) = 8$

$u(t) = c_1e^{2t} + c_2e^{-t}$

212 Homework #12 key cont'd

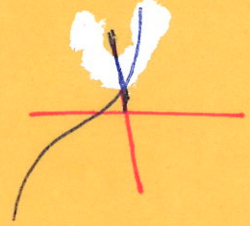
2a cont'd

$u'(t) = 2c_1e^{2t} + (-1)c_2e^{-t}$

$$\begin{aligned} c_1 + c_2 &= 3 \\ 2c_1 - c_2 &= 8 \\ \hline 3c_1 &= 11 \\ c_1 &= 11/3 \end{aligned}$$

$c_2 = 3 - 11/3 = -2/3$

$u(t) = 11/3\, e^{2t} + -2/3\, e^{-t}$



2b. $x_1' = 2x_2 \Rightarrow \frac{1}{2}x_1' = x_2 \Rightarrow \frac{1}{2}x_1'' = x_2'$

$x_2' = -2x_1$

$u(0) = 3$

$u'(0) = 2(4) = 8$

$2\left(\frac{1}{2}x_1'' = -2x_1\right) \Rightarrow x_1'' = -4x_1 \Rightarrow x_1'' + 4x_1 = 0$

$u'' + 4u = 0$

$r^2 + 4 = 0$

$r = \pm 2i$

$u(t) = c_1\cos 2t + c_2 \sin 2t \rightarrow c_1(1) + c_2(0) = 3$

$c_1 = 3$

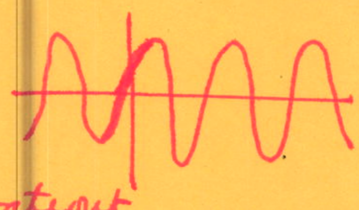
$u'(t) = -6\sin 2t + 2c_2\cos 2t \Rightarrow -6(0) + 2c_2(1) = 8$

$c_2 = 4$

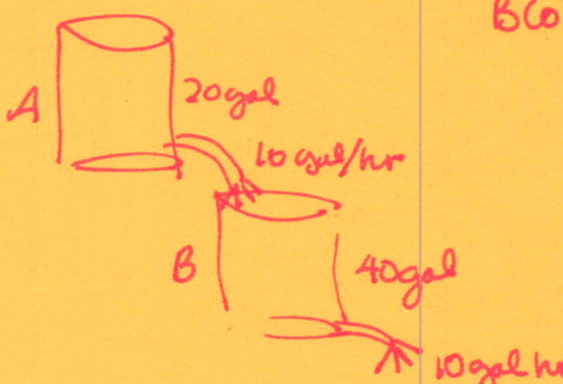
$u(t) = 3\cos 2t + 4\sin 2t$

$A(0) = 500$

$B(0) = 500$



3.



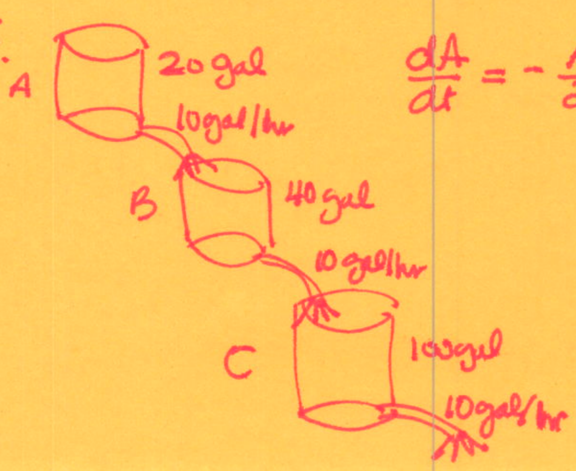
$\frac{dA}{dt} = \text{rate in} - \text{rate out}$

$\frac{dA}{dt} = 0 - \frac{A}{20}\cdot\frac{10}{hr} = -\frac{A}{2}$

$\frac{dB}{dt} = \frac{A}{2} - \frac{B}{40}\cdot\frac{10}{hr} = \frac{A}{2} - \frac{B}{4}$

$$\begin{bmatrix} A \\ B \end{bmatrix}' = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1/4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

4.



$\frac{dA}{dt} = -\frac{A}{2}$ $\quad \frac{dB}{dt} = \frac{A}{2} - \frac{B}{4}$ $\quad \frac{dC}{dt} = \frac{B}{4} - \frac{C}{100}\cdot\frac{10}{hr}$

$- \frac{C}{10}$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}' = \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/10 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

5. $\vec{x} = \begin{pmatrix}4\\2\end{pmatrix}e^{2t}$ $\quad \vec{x}' = \begin{pmatrix}3&-2\\2&-2\end{pmatrix}\vec{x}$

$$x' = 2\begin{pmatrix}4\\2\end{pmatrix}e^{2t} = \begin{pmatrix}8\\4\end{pmatrix}e^{2t} \qquad \begin{pmatrix}3&-2\\2&-2\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}e^{2t} = \begin{pmatrix}12-4\\8-4\end{pmatrix}e^{2t} = \begin{pmatrix}8\\4\end{pmatrix}e^{2t}$$

✓

6. $\vec{x} = \begin{pmatrix}6\\-8\\-4\end{pmatrix}e^{-t} + 2\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t}$ $\quad \vec{x}' = \begin{pmatrix}1&1&1\\2&1&-1\\0&-1&1\end{pmatrix}\vec{x}$

$$\vec{x}' = \begin{pmatrix}-6\\8\\4\end{pmatrix}e^{-t} + \begin{pmatrix}0\\4\\-4\end{pmatrix}e^{2t} \qquad \vec{x}' = \begin{pmatrix}1&1&1\\2&1&-1\\0&-1&1\end{pmatrix}\left[\begin{pmatrix}6\\-8\\-4\end{pmatrix}e^{-t} + 2\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t}\right]$$

$$= \begin{pmatrix}1&1&1\\2&1&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}6\\-8\\-4\end{pmatrix}e^{-t} + 2\begin{pmatrix}1&1&1\\2&1&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t} =$$

$$\begin{pmatrix}6-8-4\\12-8+4\\0\ 8-4\end{pmatrix}e^{-t} + 2\begin{pmatrix}0+1-1\\0+1+1\\0-1-1\end{pmatrix}e^{2t} = \begin{pmatrix}-6\\8\\4\end{pmatrix}e^{-t} + \begin{pmatrix}0\\4\\-4\end{pmatrix}e^{2t}$$

✓

7. $\Psi = \begin{pmatrix}e^{-3t} & e^{2t}\\ -4e^{-3t} & e^{2t}\end{pmatrix}$ $\quad \Psi' = \begin{pmatrix}1&1\\4&-2\end{pmatrix}\Psi$ $\quad \Psi' = \begin{pmatrix}-3e^{-3t} & 2e^{2t}\\ 12e^{-3t} & 2e^{2t}\end{pmatrix}$

$$\begin{pmatrix}1&1\\4&-2\end{pmatrix}\begin{pmatrix}e^{-3t} & e^{2t}\\ -4e^{-3t} & e^{2t}\end{pmatrix} = \begin{pmatrix}e^{-3t}-4e^{-3t} & e^{2t}+e^{2t}\\ 4e^{-3t}+8e^{-3t} & 4e^{2t}-2e^{2t}\end{pmatrix} = \begin{pmatrix}-3e^{-3t} & 2e^{2t}\\ 12e^{-3t} & 2e^{2t}\end{pmatrix}$$

✓

8. a. $\vec{x}' = \begin{bmatrix}1&-2\\3&-4\end{bmatrix}\vec{x}$

$(1-\lambda)(-4-\lambda)+6=0$

$\lambda^2+3\lambda-4+6=0$

$\lambda^2+3\lambda+2=0$

$(\lambda+2)(\lambda+1)=0$

$\lambda=-2, \lambda=-1$

$\begin{bmatrix}3&-2\\3&-2\end{bmatrix}$ $\quad \dfrac{3x_1=2x_2}{3}$

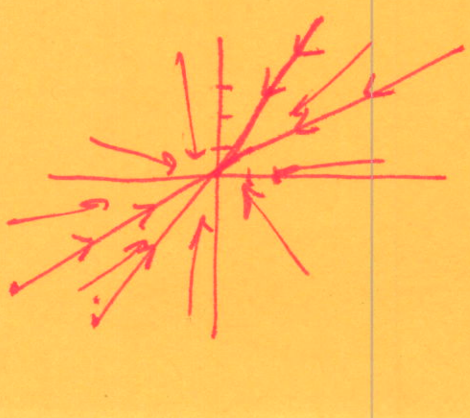
$x_1 = \frac{2}{3}x_2$ $\quad \vec{v}_1 = \begin{bmatrix}2\\3\end{bmatrix}$

$x_2 = x_2$

$\begin{bmatrix}2&-2\\3&-3\end{bmatrix}$ $\quad \dfrac{2x_1=2x_2}{2}$

$x_1 = x_2$ $\quad \vec{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

$x_2 = x_2$



origin is an attractor

$$\vec{x} = c_1\begin{bmatrix}2\\3\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-t}$$

8b. $t\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}\vec{x}$

$(4-\lambda)(-6-\lambda)+24=0$

$\lambda^2+2\lambda-24+24=0$

$\lambda^2+2\lambda=0$

$\lambda(\lambda+2)=0$

$\lambda=0 \quad \lambda=-2$

$\dfrac{4x_1=3x_2}{4}$

$x_1=\frac{3}{4}x_2$

$x_2=x_2$

$\vec{v}_1=\begin{bmatrix}3\\4\end{bmatrix}$

$\begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \div 3$

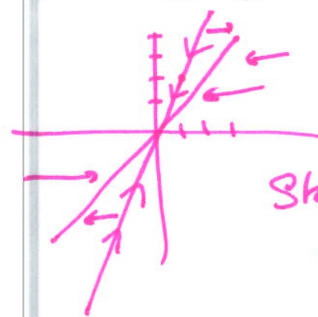
$\dfrac{2x_1=x_2}{2}$

$x_1=1/2\,x_2$

$x_2=x_2$

$\vec{v}_2=\begin{bmatrix}1\\2\end{bmatrix}$

$\vec{x}=c_1\begin{bmatrix}3\\4\end{bmatrix}t^0+c_2\begin{bmatrix}1\\2\end{bmatrix}t^{-2} \Rightarrow c_1\begin{bmatrix}3\\4\end{bmatrix}+c_2\begin{bmatrix}1\\2\end{bmatrix}t^{-2}$



Stable vector attracts

8c. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}\vec{x}$

$\lambda_1=2i \quad \lambda_2=-2i$

$\vec{v}_1=\begin{pmatrix}1/4-1/4i\\1\end{pmatrix}$ or $\begin{pmatrix}1-i\\4\end{pmatrix}$ $\quad v_2=\begin{pmatrix}1+i\\4\end{pmatrix}$

$\begin{pmatrix}1-i\\4\end{pmatrix}e^{2it}=\begin{pmatrix}1-i\\4\end{pmatrix}(\cos 2t+i\sin 2t)=\begin{pmatrix}\cos 2t-i\cos 2t+i\sin 2t+\sin 2t\\4\cos 2t+4i\sin 2t\end{pmatrix}$

$Re\begin{pmatrix}1-i\\4\end{pmatrix}e^{2it}=\begin{pmatrix}\cos 2t+\sin 2t\\4\cos 2t\end{pmatrix} \quad Im\begin{pmatrix}1-i\\4\end{pmatrix}e^{2it}=\begin{pmatrix}-\cos 2t+\sin 2t\\4\sin 2t\end{pmatrix}$

$\vec{x}=c_1\begin{pmatrix}\cos 2t+\sin 2t\\4\cos 2t\end{pmatrix}+c_2\begin{pmatrix}-\cos 2t+\sin 2t\\4\sin 2t\end{pmatrix}$

Stable orbit

d. $t\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x}$

$\lambda_1=-1, \quad \lambda_2=1$

$v_1=\begin{bmatrix}1\\3\end{bmatrix} \quad v_2=\begin{bmatrix}1\\1\end{bmatrix}$

$\vec{x}(t)=c_1\begin{bmatrix}1\\3\end{bmatrix}t^{-1}+c_2\begin{bmatrix}1\\1\end{bmatrix}t$

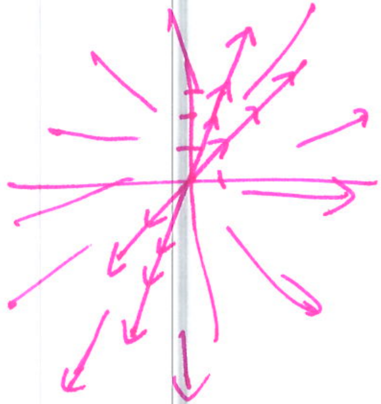
origin is saddle point

8e. $\vec{x}' = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix} \vec{x}$

$\lambda_1 = 6 \quad \lambda_2 = 4$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}$



origin repels

f. $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{x}$

$\lambda_1 = 1+2i \quad \vec{v}_1 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

$\lambda_2 = 1-2i \quad \vec{v}_2 = \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1+i \\ 2 \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 1+i \\ 2 \end{pmatrix} e^{t} (\cos 2t + i \sin 2t)$

$e^t \begin{pmatrix} \cos 2t + i\sin 2t + i\cos 2t - \sin 2t \\ 2\cos 2t + 2i\sin 2t \end{pmatrix}$

Re: $e^t \begin{pmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{pmatrix}$ Im: $e^t \begin{pmatrix} \sin 2t + \cos 2t \\ 2\sin 2t \end{pmatrix}$

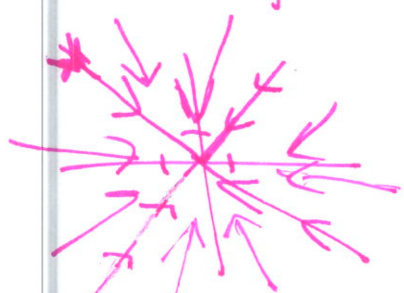
$\vec{x}(t) = c_1 \begin{pmatrix} e^t \cos 2t - e^t \sin 2t \\ 2e^t \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} e^t \sin 2t + e^t \cos 2t \\ 2e^t \sin 2t \end{pmatrix}$

Spirals outward

g. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x}$ $\quad \lambda_1 = -3 \quad \lambda_2 = -1$

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$



Origin attracts

h. $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \vec{x}$

$\lambda_1 = 1+3i \quad e^t \begin{pmatrix} 1+i \\ 2 \end{pmatrix} (\cos 3t + i\sin 3t)$

$\lambda_2 = 1-3i$

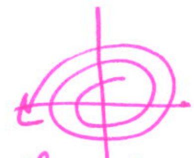
$\vec{v}_1 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$ $= e^t \begin{pmatrix} \cos 3t + i\sin 3t + i\cos 3t - \sin 3t \\ 2\cos 3t + 2i\sin 3t \end{pmatrix}$

Re: $e^t \begin{pmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{pmatrix}$ Im: $e^t \begin{pmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{pmatrix}$

8h cont'd

$$\vec{x} = c_1 e^{t}\begin{pmatrix}\cos 3t - \sin 3t \\ 2\cos 3t\end{pmatrix} + c_2 e^{t}\begin{pmatrix}\sin 3t + \cos 3t \\ 2\sin 3t\end{pmatrix}$$



Spirals outward
origin repels

8i. $\vec{x}' = \begin{bmatrix}1 & -5\\ 1 & -3\end{bmatrix}\vec{x}$ $\lambda_1 = -1+i$, $\lambda_2 = -1-i$, $\vec{v}_1 = \begin{pmatrix}2+i\\1\end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix}2-i\\1\end{pmatrix}$

$$\begin{pmatrix}2+i\\1\end{pmatrix}e^{-t}(\cos t + i\sin t) = e^{-t}\begin{pmatrix}2\cos t + 2i\sin t + i\cos t - \sin t\\ \cos t + i\sin t\end{pmatrix}$$

$$\vec{x}(t) = c_1\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix}e^{-t} + c_2\begin{pmatrix}2\sin t + \cos t\\ \sin t\end{pmatrix}e^{-t}$$

Spirals inward
Origin attracts

9a. $\vec{x}' = \begin{bmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{bmatrix}\vec{x}$ $\lambda_1 = 8$ $\lambda_2 = -1$ $\lambda_3 = -1$
$\vec{v}_1 = \begin{bmatrix}2\\1\\2\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$

$$\vec{x}(t) = c_1\begin{bmatrix}2\\1\\2\end{bmatrix}e^{8t} + c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t} + c_3\begin{bmatrix}-1\\2\\0\end{bmatrix}e^{-t}$$

Origin is saddle point

even though the eigenvalue is repeated, the vector is not defective so no extra t multiplier needed.

for graph, see attached

b. $\vec{x}' = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & -2\\ 3 & 2 & 1\end{bmatrix}\vec{x}$ $\lambda_1 = 1$ $\lambda_{2,3} = 1 \pm 2i$
$\vec{v}_1 = \begin{pmatrix}2\\-3\\2\end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix}0\\+i\\1\end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix}0\\-i\\1\end{pmatrix}$

$$\begin{pmatrix}0\\i\\1\end{pmatrix}e^{t}(\cos 2t + i\sin 2t) = e^{t}\begin{pmatrix}0\\ i\cos 2t - \sin 2t\\ \cos 2t + i\sin 2t\end{pmatrix} \Rightarrow \text{Re: } e^{t}\begin{pmatrix}0\\-\sin 2t\\ \cos 2t\end{pmatrix} \quad \text{Im: } e^{t}\begin{pmatrix}0\\ \cos 2t\\ \sin 2t\end{pmatrix}$$

$$\vec{x}(t) = c_1 e^{t}\begin{pmatrix}2\\-3\\2\end{pmatrix} + c_2 e^{t}\begin{pmatrix}0\\-\sin 2t\\ \cos 2t\end{pmatrix} + c_3 e^{t}\begin{pmatrix}0\\ \cos 2t\\ \sin 2t\end{pmatrix}$$

origin repels.

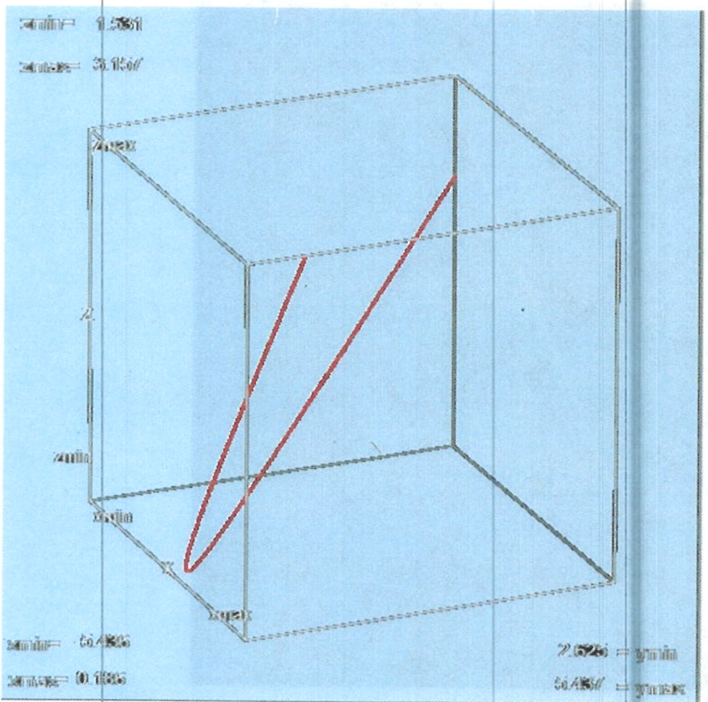
for graph, see attached

9c. $\vec{x}' = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix} \vec{x}$ $\lambda_1 = -2$ $\lambda_2 = -1$ $\lambda_3 = 1$

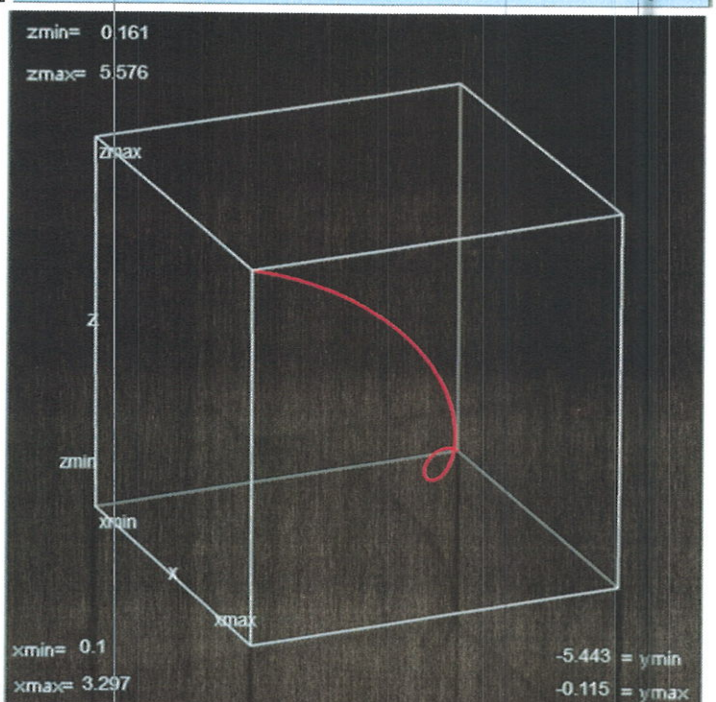
$\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}$

$\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix} e^{t}$ Origin is a saddle point

for graph, see attached

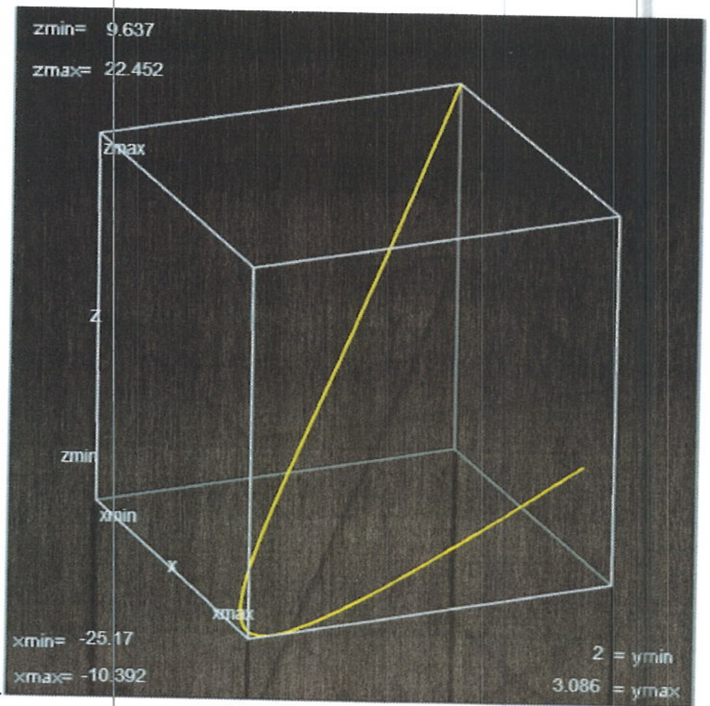


9a. c1=c2=c3=1



9b. c1=c2=c3=1

9c. c1=c2=c3=1


zmin= 9.637
zmax= 22.452
zmax
z
zmin
xmin
x
xmax
xmin= -25.17
xmax= -10.392
2 = ymin
3.086 = ymax