MAT 212, Exam #3, Summer 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Use the definition of the Laplace transform $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ to find $\mathscr{L}\{\cosh 5t\}$. (8 points)

$$\frac{1}{2}\int_0^\infty e^{-st}(e^{5t}+e^{-5t})dt = \frac{1}{2}\int_0^\infty e^{-t(s-5)}+e^{-t(s+5)}dt = \frac{1}{2}\left[\frac{-1}{s-5}e^{-t(s-5)}+\frac{-1}{s+5}e^{-t(s+5)}\right]_0^\infty$$

$$= \frac{1}{2}\left[0+\frac{1}{s-5}-0+\frac{1}{s+5}\right] = \frac{1}{2}\left[\frac{s+\cancel{5}+s-\cancel{5}}{s^2-25}\right] = \cancel{\frac{1}{2}}\left[\frac{\cancel{2}s}{s^2-25}\right] =$$

$$\boxed{\frac{s}{s^2-25}}$$

2. Use the table of Laplace transforms to find the transforms of the following functions. (4 points each)

a. $\cos^2 2t$ $= \frac{1}{2}+\frac{1}{2}\cos 4t$

$$\Rightarrow \frac{1}{2s^2}+\frac{1}{2}\cdot\frac{s}{s^2+16} = \frac{1}{2s^2}+\frac{s}{2(s^2+16)}$$

b. te^t

$$\Rightarrow \frac{1}{(s-1)^2}$$

c. $(1+t)^3$ $= t^3+3t^2+3t+1$

$$\Rightarrow \frac{6}{s^4}+\frac{3\cdot 2}{s^3}+\frac{3\cdot 1}{s^2}+\frac{1}{s} = \frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s}$$

d. $\delta(t-2)$

$$\Rightarrow e^{-2s}$$

e. $\int_0^t (t-\tau)^2 \cos\tau \, d\tau$

$f(t) = t^2 \quad g(t) = \cos t$

$\Rightarrow \frac{2}{s^3} \cdot \frac{s}{s^2+1} \doteq \frac{2}{s^2(s^2+1)}$

f. $\begin{cases} \sin t, & t < \pi \\ t & t \geq \pi \end{cases}$

$\sin t + (t - \sin t)\, u(t-\pi)$

$\sin t + \left[(t-\pi) - \pi + \sin(t-\pi)\right] u(t-\pi)$

$\Rightarrow \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{1}{s^2} - \frac{\pi}{s} + \frac{1}{s^2+1}\right]$

3. Use the table of Laplace transforms to find the inverse Laplace transform of the following function. (4 points each)

a. $\frac{5-3s}{s^2+9} = \frac{5}{s^2+9} - \frac{3s}{s^2+9} = \frac{5}{3}\left[\frac{3}{s^2+9}\right] - 3\left[\frac{s}{s^2+9}\right]$

$\Rightarrow = \frac{5}{3}\sin 3t - 3\cos 3t$

b. $\frac{3}{s^4} = \frac{1}{2}\left[\frac{3!}{s^4}\right] = \frac{1}{2}t^3$

c. $\frac{e^{-2s}(s-1)}{(s-1)^2+4} = e^{-2s}\left[\frac{s-1}{(s-1)^2+4}\right] = u(t-2)\left[e^{(t-2)}\cos(2(t-2))\right]$

$e^t \cos 2t$

d. $\frac{1}{s^2(s^2+1)}$ $= \frac{1}{s^2} \cdot \frac{1}{s^2+1}$

$\Rightarrow \int_0^t (t-\tau)\sin\tau \, d\tau$

e. $\frac{1}{s^3+3s^2+2s}$ $= \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+2)(s+1)}$ $\quad \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$

$\frac{As^2+3As+2A+Bs^2+Bs+Cs^2+2Cs}{s^3+3s^2+2s}$

$A+B+C=0$

$3A+B+2C=0$ $\quad A=1/2$

$2A=1$

$B=1/2, \; C=-1$

$\Rightarrow \frac{1}{2}+\frac{1}{2}e^{-2t}-e^{-t}$

4. Write the function $f(t) = \begin{cases} 2, & t<0 \\ t, & 0 \le t < \pi \\ t^3 \sin t, & t \ge \pi \end{cases}$ in terms of the unit step function. (8 points)

$2+(t-2)u(t\quad)+(t^3\sin t-t)\,u(t-\pi)$

5. Use Laplace transforms to solve the IVP $y''+2y'+2y=2\delta(t-2)-\delta(t-3), y(0)=1,$ $y'(0)=2$. (12 points)

$s^2Y(s)-s-2+2(sY(s)-1)+2Y(s)=2e^{-2s}-e^{-3s}$

$(s^2+2s+2)Y(s)-s-4=2e^{-2s}-e^{-3s}$

$Y(s)=\frac{2e^{-2s}-e^{-3s}+s+4}{\substack{s^2+2s+2 \\ (s+1)^2+1}} = \frac{2e^{-2s}-e^{-3s}+(s+1)+3}{(s^2+1)^2+1}$

$y(t)=3e^{-t}\sin t+e^{-t}\cos t+2e^{-(t-2)}\sin(t-2)\,u(t-2)-e^{-(t-3)}\sin(t-3)\cdot u(t-3)$

6. Write x^3 as a power series in terms of $x-2$. (6 points)

$x^3 \quad (2)=8$

$3x^2 \quad (2) = \quad 12$

$6x \quad (2) = 12$

$6 \quad (2) = 6$

$$8+12(x-2)+6(x-2)^2+(x-2)^3$$

7. Write $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n$ as a single sum. (6 points)

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n\,x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$\sum_{n=1}^{\infty} \left[a_{n+1}(n+1)n + 2a_n\right]x^n + 2a_0$$

8. For each equation, identify any singular points and classify them as regular or irregular. (6 points each)

a. $\dfrac{3x^3y'' + 2x^2y' + (1-x^2)y = 0}{3x^3}$

$$y'' + \frac{2}{3x}y' + \frac{(1-x^2)}{3x^3}y = 0$$

$x=0$ singular point

$\lim_{x\to 0} \frac{2}{3}\cdot\frac{1}{\cancel{x}}\cdot\cancel{x} \Rightarrow$ defined

$\lim_{x\to 0} \frac{(1-x^2)}{3x^{\cancel{3}}}\cdot\cancel{x^2} = \lim_{x\to 0} \frac{(1-x^2)}{3x}$ undefined

$\boxed{x=0 \text{ irregular}}$

b. $(6x^2+2x^3)y'' + 21xy' + 9(x^2-1)y = 0$

$6x^2+2x^3 = 2x^2(3+x)$

$x=0,\ x=-3$ singular points

$$y'' + \frac{21x}{2x^2(x+3)}y' + \frac{9(x^2-1)}{2x^2(3+x)}y = 0$$

$\lim_{x\to -3} \frac{21x\,\cancel{(x+3)}}{2x^2\cancel{(x+3)}}$ defined

$\lim_{x\to 0} \frac{21\cancel{x}}{2x^{\cancel{2}}(x+3)}\cdot\cancel{x}$ = defined $\quad \lim_{x\to 0} \frac{9(x^2-1)}{\cancel{2x^2}(3+x)}\cancel{x^2}$ defined $\quad \lim_{x\to -3} \frac{9(x^2-1)(x+3)^{\cancel{2}}}{2x^2\cancel{(3+x)}}$ defined

$\boxed{x=0 \text{ regular}}$ $\quad \boxed{x=-3 \text{ regular}}$

c. $\dfrac{xy'' + (x-x^3)y' + (\sin x)y = 0}{x}$

$x=0$ singular

$$y'' + (1-x^2)y + \frac{\sin x}{x}y = 0$$

$\boxed{x=0 \text{ regular}}$

$\lim_{x\to 0} (1-x^2)\cdot x$ = defined $\quad \lim_{x\to 0} \frac{\sin x}{\cancel{x}}\cdot x^{\cancel{2}}$ = defined

9. Use series solution methods to find the solution to $(x-1)y''' + 6x^2y' = 0$. State at least 4 terms of each solution (unless it is finite). Be sure that you find three solutions. (15 points)

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$

$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \qquad y''' = \sum_{n=3}^{\infty} a_n n(n-1)(n-2)x^{n-3}$

$(x-1)\sum_{n=3}^{\infty} a_n n(n-1)(n-2)x^{n-3} + 6x^2 \sum_{n=1}^{\infty} a_n n x^{n-1} = 0$

$\sum_{n=3}^{\infty} a_n n(n-1)(n-2)x^{n-2} - \sum_{n=3}^{\infty} a_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} 6a_n n x^{n+1} = 0$

$\sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)n x^n - \sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)(n+1)x^n + \sum_{n=2}^{\infty} 6a_{n-1}(n-1)x^n = 0$

$a_3(3)(2)(1)x + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)n x^n - a_3(3)(2)(1)(1) - a_4(4)(3)(2)x - \sum_{n=2}^{\infty} a_{n+3}(n+3)(n+2)(n+1)x^n$

$+ \sum_{n=2}^{\infty} 6a_{n-1}(n-1)x^n = 0$

$\sum_{n=2}^{\infty}\left[a_{n+2}(n+2)(n+1)n - a_{n+3}(n+3)(n+2)(n+1) + 6a_{n-1}(n-1)\right]x^n = 0$

a_0, a_1, a_2

$6a_3 - 24a_4 = 0$

$-6a_3 = 0 \Rightarrow a_3 = 0 \Rightarrow a_4 = 0$

$a_{n+3} = \dfrac{a_{n+2}(n+2)(n+1)n + 6a_{n-1}(n-1)}{(n+3)(n+2)(n+1)} = a_{n+2}\cdot\dfrac{n}{n+3} + a_{n-1}\cdot\dfrac{n-1}{(n+3)(n+2)(n+1)}$

$n=2$: $a_5 = \cancel{a_4 \cdot \frac{2}{5}}^{\,0} + a_1 \dfrac{1}{5\cdot4\cdot3}$ $\quad \frac{1}{60}a_1$

$n=3$: $a_6 = a_5 \cdot \frac{3}{6} + a_2 \dfrac{2}{6\cdot5\cdot4} = \frac{1}{120}a_1 + \frac{1}{60}a_2$

$n=4$: $a_7 = a_6 \cdot \frac{4}{7} + \cancel{a_3 \frac{3}{7\cdot6\cdot5}}^{\,0} = \frac{1}{210}a_1 + \frac{1}{105}a_2$

$n=5$: $a_8 = a_7\left(\frac{5}{8}\right) + \cancel{a_4 \frac{4}{8\cdot7\cdot6}}^{\,0} = \frac{1}{336}a_1 + \frac{1}{168}a_2$

$y(x) = a_0 + a_1\left(x + \frac{1}{60}x^5 + \frac{1}{120}x^6 + \frac{1}{210}x^7 + \frac{1}{336}x^8 + \dots\right) + a_2\left(x^2 + \frac{1}{60}x^6 + \frac{1}{105}x^7 + \frac{1}{168}x^8 + \dots\right)$

10. Use an appropriate series solution method to find the solution to the differential equation $xy'' + 4y = x$. (15 points)

$$y=\sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} a_n x^{n+r-1}(n+r)$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}$$

$$x\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} + 4\sum_{n=0}^{\infty} a_n x^{n+r} = x$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} 4a_n x^{n+r} = x$$

$$\sum_{n=-1}^{\infty} a_{n+1}(n+r+1)(n+r)x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r} = 1$$

$$a_0(r)(r-1)x^{r-1} + \sum_{n=0}^{\infty} a_{n+1}(n+r+1)(n+r)x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r} = \cancel{x}$$

$r=0, r=1$

$r=0$ $$\sum_{n=0}^{\infty} a_{n+1}(n+1)(n)x^n + \sum_{n=0}^{\infty} 4a_n x^n = x \Rightarrow \sum_{n=0}^{\infty}\left[a_{n+1}(n+1)n + 4a_n\right]x^n = x$$

$$a_{n+1} = \frac{-4a_n}{n(n+1)}$$

$n=0$ $\cancel{a_1(1)(0)} + 4a_0 = 0 \Rightarrow a_0 = 0$

$n=1$ $a_2(2)(1) + 4a_1 = 1 \Rightarrow \cancel{2}a_2 = -\frac{1}{2}a_1 + 1/2$

$n=2$ $$a_3 = -\frac{4a_2}{2(3)} = \frac{-2}{3}\left(-\frac{1}{2}a_1 + 1/2\right) = 1/3\, a_1 - 1/3$$

$n=3$ $$a_4 = \frac{-4a_3}{4(3)} = -\frac{1}{3}\left(1/3\, a_1 - 1/3\right) = -\frac{1}{9}a_1 + \frac{1}{9}$$

$r=1$ $$\sum_{n=0}^{\infty} a_{n+1}(n+2)(n+1)x^{n+1} + \sum_{n=0}^{\infty} 4a_n x^{n+1} = x = \sum_{n=0}^{\infty}\left[a_{n+1}(n+2)(n+1) + 4a_n\right]x^{n+1} = x$$

$n=0$ $a_1(2)(1) + 4a_0 = 0$

$a_1 = -\frac{1}{2}a_0$

$$a_{n+1} = \frac{-4a_n}{(n+2)(n+1)}$$

$n=1$ $$a_2 = \frac{-4a_1}{3\cdot 2} =$$

Same as above (shifted by x^1)

$$y(x) = c_1\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{9}x^4 + \cdots\right) + c_2 x\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{9}x^4 + \cdots\right) +$$

$$\left(1/2\, x^2 - \frac{1}{3}x^3 + \frac{1}{9}x^4 + \cdots\right)$$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \qquad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2+\omega^2}$
$\frac{1}{2}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
e^{at}	$\frac{1}{s-a} \qquad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2} \qquad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2-\omega^2} \qquad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1+at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}} \qquad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \qquad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}} \qquad s > a$	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \qquad s > 0$	$f(t - t_1)$	$e^{-t_1 s}F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \qquad$ *all s*
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2f}{df^2}$	$s^2F(s) - sf(0) - f'(0)$
$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$		