MAT 212, Exam #2, Summer 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. The Cauchy-Euler equation $x^2y'' + 2xy' - 6y = 0$ has he solution $y = x^2, y = x^{-3}$. Use the Wronskian to show that these solutions are linearly independent and form a fundamental set. Then use Abel's theorem on the original equation to confirm the value of the Wronskian (up to a constant multiple). (12 points)

$y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$

$y = x^2$ $\quad x^2(2) + 2x(2x) - 6x^2$
$y' = 2x$ $\quad = 2x^2 + 4x^2 - 6x^2 = 0$ ✓
$y'' = 2$

$y = x^{-3}$ $\quad x^2(12x^{-5}) + 2x(-3x^{-4}) - 6x^{-3}$
$y' = -3x^{-4}$ $\quad 12x^{-3} - 6x^{-3} - 6x^{-3} = 0$ ✓
$y'' = 12x^{-5}$

$W = \begin{vmatrix} x^2 & x^{-3} \\ 2x & -3x^{-4} \end{vmatrix} = -3x^{-2} - 2x^{-2} = -5x^{-2}$ ✓ fundamental set

$W = ce^{-\int \frac{2}{x}dx} = ce^{-2\ln x} = ce^{\ln x^{-2}} = cx^{-2}$ ✓

2. Find the general solution to the ODEs using the characteristic (or auxiliary) equation. (9 points each)
   a. $4y'' + 4y' + y = 0$

$4r^2 + 4r + 1 = 0$
$(2r+1)^2 = 0$
$r = -1/2$ repeated

$y(t) = c_1 e^{-1/2 t} + c_2 t e^{-1/2 t}$

b. $4y'' + 8y' + 3y = 0$

$4r^2 + 8r + 3 = 0$

$(2r+3)(2r+1) = 0$

$2r+3=0 \quad 2r+1=0$

$r = -3/2 \quad r = -1/2$

$y(t) = c_1 e^{-3/2 t} + c_2 e^{-1/2 t}$

c. $x^2y'' + xy' + y = 0$

$y = x^n$

$n(n-1) + n + 1$

$n^2 - \cancel{n} + \cancel{n} + 1 = n^2 + 1 = 0$

$n = \pm i$

$t^i = \cos(\ln t) + i \sin(\ln t)$

$= e^{(\ln t) i}$

$y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t)$

d. $y^{IV} + y = 0$

$r^4 + 1 = 0$ $\qquad r^4 = -1$

$r^2 = \pm i$

$\left(-1 = e^{\pi i} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i}\right)^{1/4}$

$r = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$

$\cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$\cos(3\pi/4) + i\sin(3\pi/4) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$\cos(5\pi/4) + i\sin(5\pi/4) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

$\cos(7\pi/4) + i\sin(7\pi/4) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

$r = \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$

$y(t) = c_1 e^{1/\sqrt{2} t} \cos(\frac{1}{\sqrt{2}} t) + c_2 e^{1/\sqrt{2} t} \sin(1/\sqrt{2} t)$
$+ c_3 e^{-1/\sqrt{2} t} \cos(\frac{1}{\sqrt{2}} t) + c_4 e^{\frac{1}{\sqrt{2}} t} \sin(\frac{1}{\sqrt{2}} t)$

3. The table below gives the solution to the second order constant coefficient homogeneous equation, and the forcing function. Determine the Ansatz for the method of undetermined coefficients in each case. (4 points each)

	y_1	y_2	y_3	$g(t)$	Ansatz
a.	$\sin 2t$	$\cos 2t$	NA	$12t$	$At + B$
b.	e^t	e^{2t}	e^{3t}	$\sin t + e^t$	$A\sin t + B\cos t + Cte^t$
c.	x	e^x	xe^x	x^3e^x	$(e^x)(Ax^3+Bx^2+Cx+D)x^2 =$
d.	$e^{-t}\sin t$	$e^{-t}\cos t$	NA	$\cosh t$	$A\cosh t + B\sinh t$

c. (multiplied out)

$Ax^5e^x + Bx^4e^x + Ce^xx^3 + De^xx^2$

4. Is it possible to use the method of undetermined coefficients for $g(t) = \cos^2 t$? Why or why not? [Hint: There is more than one way to do this. Consider taking derivatives to see if the set is finite or using an identity.] You must show work to receive credit! (8 points)

yes. method 1: $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$ both of these can be used w/ undetermined coeff. so $\cos^2 t$ can be

method 2: $\cos^2 t$, $-2\cos t \sin t$, $-2\cos^2 t + 2\sin^2 t$, etc. more of same

derivatives all consist of $A\cos^2 t + B\sin t\cos t + C\sin^2 t$

So yes, since the list is finite, we can use this method

5. The general solution to $y'' - 2y' + 2y = 0$ is $y = c_1e^x\cos x + c_2e^x\sin x$. Use that information to find the particular solution to $y'' - 2y' + 2y = x^2 + 1$. Be sure to find all the coefficients for $Y(x)$. (8 points)

$Y(x) = Ax^2 + Bx + C$

$Y'(x) = 2Ax + B$

$Y''(x) = 2A$

$Y(x) = \frac{1}{2}x^2 + x + 1$

$y(x) = c_1e^x\cos x + c_2e^x\sin x + \frac{1}{2}x^2 + x + 1$

$2A - 2(2Ax + B) + 2(Ax^2 + Bx + C)$

$2A - 4Ax - 2B + 2Ax^2 + 2Bx + 2C = x^2 + 1$

$2A = 1 \qquad A = \frac{1}{2}$

$-4A + 2B = 0 \Rightarrow -2 + 2B = 0 \Rightarrow B = 1$

$2A - 2B + 2C = 1 \quad 1 - 2 + 2C = 1 \Rightarrow -1 + 2C = 1 \Rightarrow 2C = 2 \Rightarrow C = 1$

6. Use the method of reduction of order to solve $(x+1)y'' - (x+2)y' + y = 0$, given $y_1(x) = e^x$. (12 points)

$y_2 = ve^x \quad y' = v'e^x + ve^x, \quad y'' = v''e^x + 2v'e^x + ve^x$

$(x+1)(v''e^x + 2v'e^x + ve^x) - (x+2)(v'e^x + ve^x) + ve^x = 0$

$v''xe^x + 2v'xe^x + \cancel{vxe^x} + v''e^x + \cancel{2v'e^x} + \cancel{ve^x} - v'xe^x - \cancel{vxe^x} - \cancel{2v'e^x} - \cancel{2ve^x} + \cancel{ve^x} = 0$

$v''xe^x + v''e^x + v'xe^x = 0$

$v''e^x(x+1) + v'xe^x = 0 \ / e^x$

$v''(x+1) + v'x = 0 \qquad u = v' \quad u' = v''$

$\frac{du}{dx}(x+1) = -ux$

$\int \frac{du}{u} = \int \frac{-x}{x+1}\,dx = \int -1 + \frac{1}{x+1}\,dx$

$$\begin{array}{r} -1 \\ x+1 \overline{)\,-x} \\ +x+1 \\ \hline 1 \end{array}$$

$\ln u = -x + \ln|x+1| + C = \ln e^{-x} + \ln|x+1| + \ln e^C$

$u = A(x+1)e^{-x}$

$v = \int e^{-x}(x+1)\,dx = -(x+1)e^{-x} - e^{-x} + C = -xe^{-x} - e^{-x} - e^{-x} = -(x+2)e^{-x}$

+/-	u	dv
+	x+1	e^{-x}
−	1	$-e^{-x}$
+	0	e^{-x}

$y_2 = -(x+2)e^{-x} \cdot e^x = -(x+2)$

$\boxed{y_2 = (x+2)}$

7. A mass of 3 kg is attached to the end of a spring that is stretched 20 cm by a force of 15 N. It is set in motion with initial position $x(0) = 0$ and initial velocity $x'(0) = -10$m/s. Set up the equation to solve the system. [You don't need to solve.] Is the system undamped, underdamped, critically damped or overdamped? (8 points)

$15 = k(.2) \qquad m = 3$

$k = 75$

$3x'' + 75x = 0 \qquad$ *undamped*

or $\quad x'' + 25x = 0$

8. Use the method of variation of parameters to find the particular solution to $y'' + 9y = \csc^2 3x$. [Hint: $Y(t) = -y_1 \int \frac{y_2 g}{W} dt + y_2 \int \frac{y_1 g}{W} dt$.] (10 points)

$r^2 + 9 = 0 \quad r = \pm 3i \quad c_1 \cos 3t + c_2 \sin 3t$

$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\cos^2 3t + 3\sin^2 3t = 3$

$Y(t) = -\cos 3t \int \frac{\sin 3t \csc^2 3t}{3} dt + \sin 3t \int \frac{\cos 3t \csc^2 3t}{3} dt$

$= -\frac{1}{3}\cos 3t \int \csc 3t \, dt + \frac{1}{3}\sin 3t \int \frac{\cos 3t}{\sin 3t} \cdot \frac{1}{\sin 3t} dt$

$\cot 3t \cdot \csc 3t$

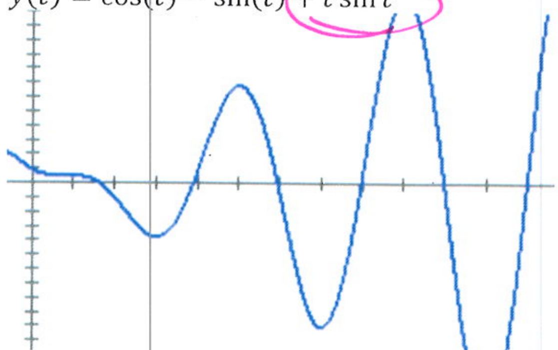
$= +\frac{1}{3}\cos 3t \ln|\csc 3t + \cot 3t| \cdot \frac{1}{3} + \frac{1}{3}\sin 3t(-\csc 3t) \cdot \frac{1}{3}$

$= \frac{1}{9}\cos 3t \cdot \ln|\csc 3t + \cot 3t| - \frac{1}{9}$

$y(x) = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{9}\cos(3x) \ln|\csc 3x + \cot 3x| - \frac{1}{9}$

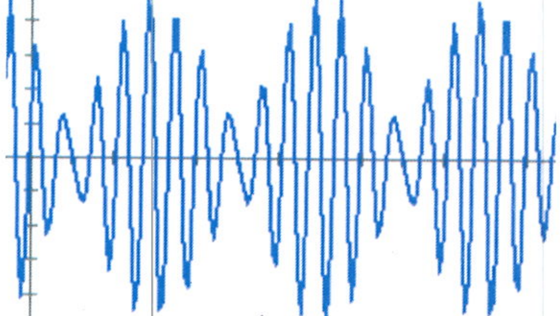
9. Below are the graphs of solutions to forced spring problems. Determine if the solution models resonance or beats (or neither). Explain your reasoning. (4 points each)

a. $y(t) = \cos(t) - \sin(t) + t\sin t$



resonance
amplitude increasing w/ time
$t\sin t$ term from forcing function w/ same frequency as natural frequency

b. $y(t) = 3\sin 6t + 2\cos(7t)$



beats
2 trig functions one natural one similar but not identical

10. An electrical circuit has an inductance of $L = 10H$ and a capacitance of $C = 0.02F$. Find the resistance on the circuit (in ohms) needed to achieve critical damping. (10 points)

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) = 0$$

$$10Q'' + RQ' + \underbrace{\frac{1}{.02}}_{=50}Q = 0$$

$$10Q'' + RQ' + 50Q = 0$$

$$\frac{-R \pm \sqrt{R^2 - 4(10)(50)}}{2 \cdot 10}$$

Critical damping = discriminant = 0

$$R^2 - 2000 = 0$$

$$R^2 = 2000$$

$$R \approx 44.72\ \Omega$$

$$10\sqrt{20} = 20\sqrt{5}$$

11. Sketch a graph of what an underdamped spring system looks like. (4 points)