202 Proof Set #2 key

1. We want to show that if a, b real and $1 < a < b$, then $1/a > 1/b$. Suppose it's not the case that $1 < a < b$, i.e. $0 < b < a < 1$. If $a, b \neq 0$ we can divide $b < a$ by $ab > 0$, then $\frac{b}{ab} < \frac{a}{ab} < \frac{1}{ab} \Rightarrow \frac{1}{a} < \frac{1}{b}$ therefore, if $1 < a < b$, $\frac{1}{a} > 1/b$. //

2. To show that the zero vector in a vector space is unique, we will assume it is not unique. Both $\vec{u}$ and $\vec{v}$ are the zero vector. Then we need to show that $\vec{u} = \vec{v}$. The zero vector is the additive identity so $\vec{x} + \vec{0} = \vec{x}$. If both $\vec{u}$ and $\vec{v}$ act like the zero vector, then $\vec{x} + \vec{u} = \vec{x}$ and $\vec{x} + \vec{v} = \vec{x}$. Using the property of the additive inverse, we add $-\vec{x}$ to both sides of the equation giving us $\vec{x} + \vec{v} + (-\vec{x}) = \vec{x} + (-\vec{x})$ and $\vec{x} + \vec{u} + (-\vec{x}) = \vec{x} + (-\vec{x})$ which implies $\vec{x} + (-\vec{x}) + \vec{v} = \vec{x} + (-\vec{x})$ and $\vec{x} + (-\vec{x}) + \vec{u} = \vec{x} + (-\vec{x})$ by the property of commutativity. By definition of the additive inverse we can cancel $\vec{x} + (-\vec{x}) = \vec{0}$ which implies $\vec{u} + \vec{0}$ and $\vec{v} + \vec{0} = \vec{0}$ in both cases. Thus $\vec{u} = \vec{v}$. The zero vector is not unique. //

3. a. Let $= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

if we exchange two rows of A to obtain B then $B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$

$\det A = a_{11}a_{22} - a_{21}a_{12}$ and $\det B = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{21}a_{12})$

$= -\det A$. //.

b. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then by definition of the transpose, $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$\det A^T = a_{11}a_{22} - a_{12}a_{21}$. Since a_{12}, a_{21} are real numbers, $a_{11}a_{22} - a_{12}a_{21} =$

$a_{11}a_{22} - a_{21}a_{12} = \det A$. //

c. If $AC = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22} \end{bmatrix}$ then

$\det(AC) = (a_{11}c_{11}+a_{12}c_{21})(a_{21}c_{12}+a_{22}c_{22}) - (a_{21}c_{11}+a_{22}c_{21})(a_{11}c_{12}+a_{12}c_{22}) =$

$\cancel{a_{11}c_{11}a_{21}c_{12}} + a_{11}c_{11}a_{22}c_{22} + a_{12}c_{21}a_{21}c_{12} + \cancel{a_{12}c_{21}a_{22}c_{22}} - \cancel{a_{21}c_{11}a_{11}c_{12}} -$

$a_{21}c_{11}a_{12}c_{22} - a_{22}c_{21}a_{11}c_{12} - \cancel{a_{22}c_{21}a_{12}c_{22}}$

on the other hand $\det A = (a_{11}a_{22} - a_{12}a_{21})$ and $\det C = (c_{11}c_{22} - c_{12}c_{21})$

So $\det A \cdot \det C = (a_{11}a_{22} - a_{12}a_{21})(c_{11}c_{22} - c_{12}c_{21}) =$

$a_{11}a_{22}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21}$

after cancelling terms and comparing the remaining ones we find that

$a_{11}c_{11}a_{22}c_{22} + a_{12}c_{21}a_{21}c_{12} - a_{21}c_{11}a_{12}c_{22} - a_{22}c_{21}a_{11}c_{12} =$

$a_{11}a_{22}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21}$

Thus $\det(AC) = \det A \cdot \det C$. //

d. first, it's clear from the property above that $\det A^2 = \det(A \cdot A) = \det A \cdot \det A = (\det A)^2$. So we suppose the property is true for $\det A^k = (\det A)^k$ and show true for $k+1$. $\det(A^{k+1}) = \det(A^k \cdot A) = \det(A^k) \cdot \det A = (\det A)^k \cdot \det A = (\det A)^{k+1}$. Thus $\det(A^k) = (\det A)^k$ is true for all $k \in \mathbb{N}$. //

e. Consider r is real and $rA = \begin{bmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{bmatrix}$ and $\det(rA) =$

$ra_{11}ra_{22} - ra_{12}ra_{21} = r^2a_{11}a_{22} - r^2a_{12}a_{21}$ by commutativity and associativity of real #'s and by the distributive property we have $r^2(a_{11}a_{22} - a_{12}a_{21}) = r^2 \det A$.

Thus, $\det(rA) = r^2 \det A$. //

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4. if AB is invertible then $\det(AB) \neq 0$. Since $\det(AB) = \det A \cdot \det B$, this is also $\neq 0$. but if either $\det A = 0$ or $\det B = 0$ then $\det A \cdot \det B$ would $= 0$. Thus $\det A \neq 0$ and $\det B \neq 0$. So both A and B must be invertible (as both are nonsingular). //

5. let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. $u_i, v_i, w_i, c, d \in \mathbb{R}$

a. $\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$ by property of vector addition.

using commutativity of addition of real numbers in each component, we have

$\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+u_3 \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ v_2+u_2 \\ v_3+u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \vec{v} + \vec{u}$ by definition of vector addition.

Thus $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. //

b. $(\vec{u} + \vec{v}) + \vec{w} = \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} (u_1+v_1) \\ (u_2+v_2) \\ (u_3+v_3) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} (u_1+v_1)+w_1 \\ (u_2+v_2)+w_2 \\ (u_3+v_3)+w_3 \end{bmatrix}$

Then by vector addition, ...

by associativity of real numbers applied to each component we have $\begin{bmatrix} (u_1+v_1)+w_1 \\ (u_2+v_2)+w_2 \\ (u_3+v_3)+w_3 \end{bmatrix} = \begin{bmatrix} u_1+(v_1+w_1) \\ u_2+(v_2+w_2) \\ u_3+(v_3+w_3) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \left(\begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right)$

by vector addition, and so this equals $\vec{u} + (\vec{v} + \vec{w})$. Thus $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$. //

c. for c real, $c(\vec{u} + \vec{v}) = c\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = c\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$ by vector addition,

and in turn this equals $\begin{bmatrix} c(u_1+v_1) \\ c(u_2+v_2) \\ c(u_3+v_3) \end{bmatrix}$. Using the distributive property of real numbers on each component, we get $\begin{bmatrix} cu_1+cv_1 \\ cu_2+cv_2 \\ cu_3+cv_3 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} + \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}$ by definition of vector addition and by scalar multiplication this is $c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = c\vec{u} + c\vec{v}$. Thus $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$. //

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d. for c, d real, $(c+d)\vec{u} = (c+d)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} (c+d)u_1 \\ (c+d)u_2 \\ (c+d)u_3 \end{bmatrix}$ by scalar multiplication.

using property of distribution of real numbers in each component we get

$\begin{bmatrix} cu_1 + du_1 \\ cu_2 + du_2 \\ cu_3 + du_3 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} + \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix}$ by vector addition, and so by scalar

multiplication we get $c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + d\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c\vec{a} + d\vec{u}$. Thus $(c+d)\vec{a} = c\vec{u} + d\vec{u}$. //

6. polynomial of degree at most 3 given by $p(t) = a_0 + a_1x + a_2x^2 + a_3x^3$ for a_0, a_1, a_2, a_3 real. (These are elements in P_3). P_n is a vector space, So we need show that the zero vector exists, and sum and scalar multiplication produce elements in the set P_3.

i.) suppose that $a_0 = a_1 = a_2 = a_3 = 0$ then $p(t) = 0$, which is the zero vector in P_3.

ii). let $q(t) = b_0 + b_1x + b_2x^2 + b_3x^3$. then $\vec{p}(t) + \vec{q}(t) = a_0 + a_1x + a_2x^2 +$ w/ b_0, b_1, b_2, b_3 in $\mathbb{R}$. $a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3 =$ $(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3$. since all the coefficients are real, this polynomial is in $\mathbb{P}_3$. $(p(t)+q(t) \in \mathbb{P}_3)$

iii). for k real, $kp(t) = k(a_0 + a_1x + a_2x^2 + a_3x^3) = (ka_0) + (ka_1)x + (ka_2)x^2 + (ka_3)x^3$. and since all coefficients are real $kp(t)$ in $\mathbb{P}_3$.

thus P_3 is a subspace of P_n. //

7. the set $S = \left\{ \begin{bmatrix} a & a^2 & a^3 \\ a^2 & a & a^2 \\ a^3 & a^2 & a \end{bmatrix} : a \text{ is real} \right\}$ is not a subspace of $M_{3\times3}$.

the definition of this set requires the elements $a_{12}, a_{21}, a_{32}, a_{23}$ must all be positive since $a^2 \geq 0$ for all real a.

thus $k\begin{bmatrix} a & a^2 & a \\ a^2 & a & a^2 \\ a^3 & a^2 & a \end{bmatrix}$ for $k=-1, a=1$ is not in S since $-1\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

if we take $a = -1$ (on the diagonal) the elements in $a_{12}, a_{21}, a_{23}, a_{32}$ do not equal $a^2 = 1$. //

10 cont'd.

have not included. we can do this forever, thus P_∞ is infinite dimensional.

11.a. consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

if we exchange two columns such as $C_1 \leftrightarrow C_2$, $C_2 \leftrightarrow C_3$, or $C_1 \leftrightarrow C_3$. the elementary matrices that represent these changes are $E_{12} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ respectively. The determinant of each matrix is -1. Thus since AE is the resulting matrix, the determinant of the new matrix is $\det(AE) = \det A \cdot \det E = (-1)\det A = -\det A$.

Thus the column operation of exchanging columns changes the determinant by a sign. (note that column operations multiply on the right rather than the left.)

b. consider the elementary matrices that add two columns. $E_{121} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which adds $C_1 + C_2 \to C_1$; $E_{122} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponding to $C_1 + C_2 \to C_2$, $E_{131} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ corresponding to $C_1 + C_3 \to C_1$, $E_{133} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponding to $C_1 + C_3 \to C_3$, $E_{232} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ corresponding to $C_2 + C_3 \to C_2$, and $E_{233} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ corresponding to $C_2 + C_3 \to C_3$. each of the determinants are 1, so AE is the resulting matrix from the change and $\det(AE) = \det A \cdot \det E = \det A(1) = \det A$. which is no change.

c. the elementary matrices corresponding to multiplying columns by a scalar k are $E_1 = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$ corresponding to $kC_1 \to C_1$, $kC_2 \to C_2$ and $kC_3 \to C_3$. Since these matrices are diagonal, it's clear the determinant of each is k. Thus AE is the resulting matrix and $\det(AE) = \det A \cdot \det E = \det A \cdot (k) = k \det A$. //

12. Your answers and selections of properties may differ.

$A = |A| \neq 0$

B = A reduces to The I_n. ($n \times n$ identity).

$C = A\vec{x} = \vec{0}$ has only the trivial solution.

I will show $A \to B \to C \to A$.

Let us assume that the determinant of A is zero. We will show this implies that A reduces to I_n. We do this by letting $\det A = k$ for some $k \in \mathbb{R}$, $k \neq 0$. Since $\det A = k \neq 0$, A is invertible by theorem 3.7 (in the textbook). to obtain I_n we multiply by $A^{-1}A = I_n$. Since any invertible matrix can be written as a product of elementary matrices, this applies to both A^{-1} and A, thus elementary row operations can reduce A to the identity. If A reduces to the identity then we can reduce the augmented matrix representing $A\vec{x} = \vec{0}$ to $[I|\vec{0}]$ by the same operations as above. Such an augmented matrix represents the solution $\vec{x} = \vec{0}$, which is the trivial solution. And finally if $A\vec{x} = \vec{0}$ has only the trivial solution, then the system is neither dependent nor inconsistent, meaning there must be a pivot on the diagonal of A when reduced. All pivots are non-zero and the product of a triangular (or diagonal) matrix is the product of the entries on the diagonal. Since all the factors are non-zero, the product is also non-zero, so the determinant of $A \neq 0$. //