202 Homework #10 key

1. a. $P = \begin{bmatrix} .7 & .1 & .05 \\ .2 & .5 & .2 \\ .1 & .4 & .75 \end{bmatrix}$

b. $\vec{x}_0 = \begin{bmatrix} .8 \\ .1 \\ .1 \end{bmatrix}$ $\quad \vec{x}_1 = P\vec{x}_0 = \begin{bmatrix} .575 \\ .23 \\ .195 \end{bmatrix}$

c. p^big(120) $= \begin{bmatrix} 9/49 & 9/49 & 9/49 \\ 2/7 & 2/7 & 2/7 \\ 26/49 & 26/49 & 26/49 \end{bmatrix}$ $\quad \vec{x}_\infty = \begin{bmatrix} 9/49 \\ 2/7 \\ 26/49 \end{bmatrix}$

2. $\begin{bmatrix} .75 & .70 & .10 & 0 \\ .22 & .25 & .40 & 0 \\ .027 & .045 & .45 & 0 \\ .003 & .005 & .05 & 1 \end{bmatrix}$ $\quad \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$P\vec{x}_0 = \vec{x}_1 = \begin{bmatrix} .75 \\ .22 \\ .027 \\ .003 \end{bmatrix}$ in people 750 healthy
220 ill
27 very ill
3 dead

around 500 steps or roughly 41 years (since each step is a month)

3. a. There is one equilibrium vector since There is communication between all states

$P - I = \begin{bmatrix} -.3 & .4 \\ .3 & -.4 \end{bmatrix}$ $\quad .3x_1 = .4x_2$, $x_1 = \frac{4}{3}x_2$, $x_2 = x_2$ $\quad \vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\quad 4+3=7$ $\quad \vec{q} = \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix}$

b. communication between all states; one $\vec{q}$ $\quad 11+15+21=47$

$P - I = \begin{bmatrix} -.6 & .3 & .1 \\ .3 & -.5 & .2 \\ .3 & .2 & -.3 \end{bmatrix} \xRightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -11/21 \\ 0 & 1 & -5/7 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = 11/21\, x_3$, $x_2 = 5/7\, x_3$, $x_3 = x_3$ $\quad \vec{v} = \begin{bmatrix} 11 \\ 15 \\ 21 \end{bmatrix}$ $\quad \vec{q} = \begin{bmatrix} 11/47 \\ 15/47 \\ 21/47 \end{bmatrix}$

c. there is communication, but x_2 is an absorbing state so $q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

d. there are two $\vec{q}$'s since $A \leftrightarrow D$ and $B \leftrightarrow C$ only. (basis dim = 2)

$\begin{bmatrix} -.3 & 0 & 0 & .4 \\ 0 & -.5 & .2 & 0 \\ 0 & .5 & -.2 & 0 \\ .3 & 0 & 0 & -.4 \end{bmatrix} \xRightarrow{\text{rref}}$

$.3x_1 = .4x_4$, $x_4 = x_4$, $x_2 = 0$, $x_3 = 0$, $x_1 = \frac{4}{3}x_4$ $\quad \vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \vec{q}_1 = \begin{bmatrix} 4/7 \\ 0 \\ 0 \\ 3/7 \end{bmatrix}$ $\quad 4+3=7$

$.5x_2 = .2x_3$, $x_2 = \frac{2}{5}x_3$, $x_3 = x_3$, $x_1 = 0$, $x_4 = 0$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 0 \end{bmatrix}$ $\quad \vec{q}_2 = \begin{bmatrix} 0 \\ 2/7 \\ 5/7 \\ 0 \end{bmatrix}$ $\quad 2+5=7$

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3d cont'd. any steady state can be expressed as $a\vec{q_1} + b\vec{q_2}$ where $a+b=1$ and represents the original distribution of population between the two sets of communicating states.

e. there is only one $\vec{q}$ since there is a path through all the states regardless of where one begins.

$$P-I = \begin{bmatrix} -.3 & 0 & 0 & .4 \\ .2 & -.5 & .1 & 0 \\ 0 & .5 & -.9 & 0 \\ .1 & 0 & .8 & -.4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 4/3 x_4$
$x_2 = 3/5 x_4$
$x_3 = 1/3 x_4$
$x_4 = x_4$

$$\vec{v} = \begin{bmatrix} 20 \\ 9 \\ 5 \\ 15 \end{bmatrix} \quad \vec{q} = \begin{bmatrix} 20/49 \\ 9/49 \\ 5/49 \\ 15/49 \end{bmatrix}$$

$20+9+5+15=49$

f. one $\vec{q}$

$$P-I = \begin{bmatrix} -.9 & .7 \\ .9 & -.7 \end{bmatrix} \quad \begin{matrix} x_1 = \frac{7}{9}x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \quad \vec{q} = \begin{bmatrix} 7/16 \\ 9/16 \end{bmatrix}$$

$7+9=16$

g. one $\vec{q}$

$$P-I = \begin{bmatrix} -.4 & .2 & .04 \\ .25 & -.5 & .01 \\ .15 & .3 & -.05 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -11/75 \\ 0 & 1 & -7/75 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 11/75 x_3$
$x_2 = 7/75 x_3$
$x_3 = x_3$

$$\vec{v} = \begin{bmatrix} 11 \\ 7 \\ 75 \end{bmatrix} \quad \vec{q} = \begin{bmatrix} 11/93 \\ 7/93 \\ 75/93 \end{bmatrix}$$

$11+7+75=93$

4a. false. it may or may not.

b. true. (it's for an eigenvector associated w/ $\lambda=1$)

5. a. $\begin{bmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{bmatrix}$

$(2-\lambda)(1-\lambda) - 2 = 0$ $\quad \lambda=0, \lambda=3$
$\lambda^2 - 3\lambda + 2 - 2 = 0$
$\lambda^2 - 3\lambda = 0 = \lambda(\lambda-3)$

$\lambda=0$ $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ $\quad \cancel{\sqrt{2}x_1}$ $x_1 = -\frac{\sqrt{2}}{2}x_2$, $x_2 = x_2$ $\quad \vec{v_1} = \begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix}$

$\vec{v_1} \cdot \vec{v_2} = -2 + 2 = 0$

They are orthogonal

$\lambda_2=3$ $\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$ $\quad x_1 = \sqrt{2}x_2$, $x_2 = x_2$ $\quad \vec{v_2} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 2 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

5b. $\begin{bmatrix} -\lambda & 4 & 4 \\ 4 & 2-\lambda & 0 \\ 4 & 0 & -2-\lambda \end{bmatrix} \Rightarrow 4\begin{vmatrix} 4 & (2-\lambda) \\ 4 & 0 \end{vmatrix} + (-2-\lambda)\begin{vmatrix} -\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} =$

$4\left[0 - 4(2-\lambda)\right] + (-2-\lambda)\left[(-\lambda)(2-\lambda) - 16\right] =$

$4(-8+4\lambda) - (2+\lambda)(\lambda^2 - 2\lambda - 16) =$

$-32 + 16\lambda - (2\lambda^2 - 4\lambda - 32 + \lambda^3 - 2\lambda^2 - 16\lambda)$

$\cancel{-32} + 16\lambda \cancel{- 2\lambda^2} + 4\lambda \cancel{+ 32} - \lambda^3 \cancel{+ 2\lambda^2} + 16\lambda$

$-\lambda^3 + 36\lambda = -\lambda(\lambda^2 - 36) = -\lambda(\lambda-6)(\lambda+6) = 0$

$\lambda = 0, 6, -6$

$\lambda_1 = 0$ $\begin{bmatrix} 0 & 4 & 4 \\ 4 & 2 & 0 \\ 4 & 0 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 &= 1/2 x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

$\lambda_2 = 6$ $\begin{bmatrix} -6 & 4 & 4 \\ 4 & -4 & 0 \\ 4 & 0 & -8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 &= 2x_3 \\ x_2 &= 2x_3 \\ x_3 &= x_3 \end{aligned}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$\lambda_3 = -6$ $\begin{bmatrix} 6 & 4 & 4 \\ 4 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 &= -x_3 \\ x_2 &= 1/2 x_3 \\ x_3 &= x_3 \end{aligned}$ $\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = 2 - 4 + 2 = 0$ $\quad \vec{v}_1 \cdot \vec{v}_3 = -2 - 2 + 4 = 0$ $\quad \vec{v}_2 \cdot \vec{v}_3 = -4 + 2 + 2 = 0$

c. $\begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix} \Rightarrow (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 2 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} -1 & 1-\lambda \\ 2 & 2 \end{vmatrix}$

$= (1-\lambda)\left[(1-\lambda)(2-\lambda) - 4\right] + (-2+\lambda-4) + 2(-2-2(1-\lambda))$

$= (1-\lambda)(\lambda^2 - 3\lambda + 2 - 4) + (\lambda - 6) + 2(\underbrace{-2-2}_{-4}\underbrace{+2\lambda}_{+2\lambda})$

$= (1-\lambda)(\lambda^2 - 3\lambda - 2) + (\lambda - 6) + (-8) + 4\lambda$

$= \lambda^2 - 3\cancel{\lambda} - 2 - \lambda^3 + 3\lambda^2 + \cancel{2\lambda} + \cancel{\lambda} - 6 - 8 + 4\lambda$

$= -\lambda^3 + 4\lambda^2 + 4\lambda - 16 = -\lambda^2(\lambda - 4) + 4(\lambda - 4) =$

$(\lambda-4)(4-\lambda^2) = (\lambda-4)(\lambda+2)(2-\lambda) = 0$ $\quad \lambda = 4, 2, -2$

$\lambda_1 = 4$

$\begin{bmatrix} -3 & -1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 &= 1/2 x_3 \\ x_2 &= 1/2 x_3 \\ x_3 &= x_3 \end{aligned}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

5c. Cont'd

$\lambda_2 = 2$

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \overset{\text{rref}}{\Rightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \\ x_3 = 0 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3 = -2$

$$\begin{bmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 2 & 2 & 4 \end{bmatrix} \overset{\text{rref}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{matrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\vec{v}_1 \cdot \vec{v}_2 = -1+1+0 = 0 \quad \vec{v}_1 \cdot \vec{v}_3 = -1-1+2 = 0 \quad \vec{v}_2 \cdot \vec{v}_3 = 1-1+0 = 0$

6a. $\begin{bmatrix} .38-\lambda & .24 \\ -.36 & 1.22-\lambda \end{bmatrix} \Rightarrow (.38-\lambda)(1.22-\lambda) + .0864 = \lambda^2 - 1.6\lambda + .55 = 0$

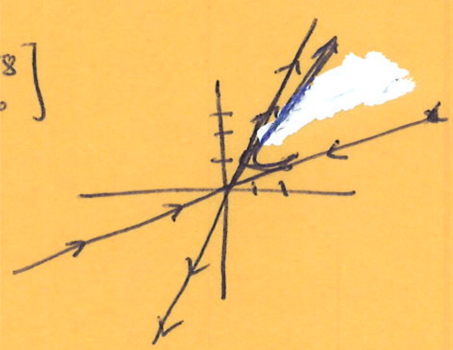
$(\lambda - 1.1)(\lambda - .5) = 0 \quad \lambda = 1.1, \ \lambda = .5$

origin is a saddle.

$\vec{x}_0 = \begin{bmatrix} 15 \\ 20 \end{bmatrix} \quad \vec{x}_1 = \begin{bmatrix} 10.5 \\ 19 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 8.55 \\ 19.4 \end{bmatrix} \quad \vec{x}_3 = \begin{bmatrix} 7.905 \\ 20.59 \end{bmatrix} \quad \vec{x}_4 = \begin{bmatrix} 7.9455 \\ 22.274 \end{bmatrix} \quad \vec{x}_5 = \begin{bmatrix} 8.365 \\ 24.31 \end{bmatrix}$

$\vec{x}_6 = \begin{bmatrix} 9.01 \\ 26.65 \end{bmatrix} \quad \vec{x}_7 = \begin{bmatrix} 9.82 \\ 29.27 \end{bmatrix} \quad \vec{x}_8 = \begin{bmatrix} 10.76 \\ 32.17 \end{bmatrix} \quad \vec{x}_9 = \begin{bmatrix} 11.81 \\ 35.38 \end{bmatrix} \quad \vec{x}_{10} = \begin{bmatrix} 12.98 \\ 38.91 \end{bmatrix} \quad \vec{x}_{11} = \begin{bmatrix} 14.27 \\ 42.8 \end{bmatrix}$

$\vec{x}_{12} = \begin{bmatrix} 15.69 \\ 47.08 \end{bmatrix} \quad \vec{x}_{13} = \begin{bmatrix} 17.26 \\ 51.78 \end{bmatrix} \quad \vec{x}_{14} = \begin{bmatrix} 18.99 \\ 56.96 \end{bmatrix} \quad \vec{x}_{15} = \begin{bmatrix} 20.88 \\ 62.66 \end{bmatrix}$



$\lambda_1 = 1.1 \quad \begin{bmatrix} \overset{=-.72}{.38-1.1} & .24 \\ -.36 & .12 \end{bmatrix} \quad \begin{matrix} x_1 = \frac{.12}{.36} x_2 \\ x_1 = \frac{1}{3} x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = .5 \quad \begin{bmatrix} -.12 & .24 \\ -.36 & .72 \end{bmatrix} \quad \begin{matrix} x_1 = \frac{.24}{.12} x_2 \\ x_1 = 2x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{x}_n = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (1.1)^n + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} (.5)^n$

6b. $\begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} \Rightarrow (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 4 + 2 = \lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0 \quad \lambda = 3, 2.$ origin repels

$\lambda_1 = 3 \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \quad \begin{matrix} x_1 = -\frac{1}{2} x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \lambda_2 = 2 \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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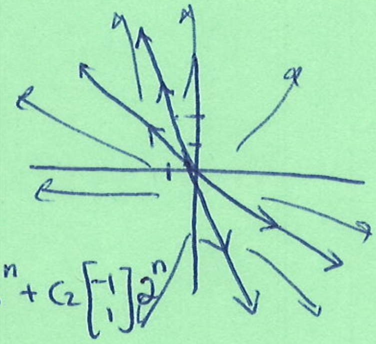
$\vec{X_0} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\vec{X_1} = \begin{bmatrix} -3 \\ 24 \end{bmatrix}$ $\vec{X_2} = \begin{bmatrix} -27 \\ 90 \end{bmatrix}$ $\vec{X_3} = \begin{bmatrix} -117 \\ 306 \end{bmatrix}$



$\vec{X_4} = \begin{bmatrix} -423 \\ 990 \end{bmatrix}$ $\vec{X_5} = \begin{bmatrix} -1413 \\ 3114 \end{bmatrix}$ they just keep getting bigger

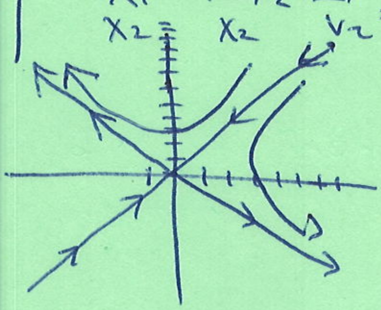
$\vec{X_n} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} 3^n + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} 2^n$

6c. $\begin{bmatrix} 1.71-\lambda & -.707 \\ 1 & -\lambda \end{bmatrix} \Rightarrow (1.71-\lambda)(-\lambda) + .707 =$

$\lambda^2 - 1.71\lambda + .707 = 0$

$\lambda_1 = .7$ $\quad \lambda = .7, \ \lambda = 1.01$ Saddle = origin

$\begin{bmatrix} 1.01 & -.707 \\ 1 & -.7 \end{bmatrix}$ $\begin{matrix} x_1 = .7x_2 \\ x_2 = x_2 \end{matrix}$ $\begin{bmatrix} 7 \\ 10 \end{bmatrix} = \vec{v_1}$ $\quad \lambda_2 = 1.01$ $\begin{bmatrix} .7 & -.707 \\ 1 & 1.01 \end{bmatrix}$ $\begin{matrix} x_1 = -1.01x_2 \\ x_2 = x_2 \end{matrix}$ $\vec{v_2} = \begin{bmatrix} -1.01 \\ 1 \end{bmatrix} = \begin{bmatrix} -101 \\ 100 \end{bmatrix}$



$\vec{X_0} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$ $\vec{X_1} = \begin{bmatrix} 9.619 \\ 11 \end{bmatrix}$ $\vec{X_2} = \begin{bmatrix} 8.67 \\ 9.62 \end{bmatrix}$ $\vec{X_3} = \begin{bmatrix} 8.03 \\ 8.67 \end{bmatrix}$

$\vec{X_4} = \begin{bmatrix} 7.60 \\ 8.03 \end{bmatrix}$ $\vec{X_5} = \begin{bmatrix} 7.31 \\ 7.6 \end{bmatrix}$ $\vec{X_6} = \begin{bmatrix} 7.14 \\ 7.31 \end{bmatrix}$ $\vec{X_7} = \begin{bmatrix} 7.03 \\ 7.18 \end{bmatrix}$

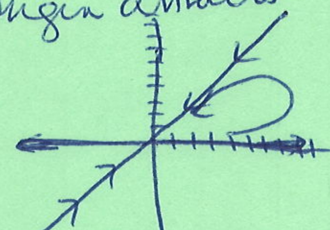
$\vec{X_8} = \begin{bmatrix} 6.98 \\ 7.03 \end{bmatrix}$ $\vec{X_9} = \begin{bmatrix} 6.96 \\ 6.98 \end{bmatrix}$ $\vec{X_{10}} = \begin{bmatrix} 6.97 \\ 6.96 \end{bmatrix}$ $\vec{X_{11}} = \begin{bmatrix} 7.00 \\ 6.97 \end{bmatrix}$ $\vec{X_{12}} = \begin{bmatrix} 7.04 \\ 7.00 \end{bmatrix}$ $\vec{X_{13}} = \begin{bmatrix} 7.09 \\ 7.04 \end{bmatrix}$

$\vec{X_{14}} = \begin{bmatrix} 7.15 \\ 7.09 \end{bmatrix}$ $\vec{X_{15}} = \begin{bmatrix} 7.21 \\ 7.14 \end{bmatrix}$ $\quad \vec{X_n} = c_1 \begin{bmatrix} 7 \\ 10 \end{bmatrix} (.7)^n + c_2 \begin{bmatrix} -101 \\ 100 \end{bmatrix} (1.01)^n$

6d. $\begin{bmatrix} 1.8-\lambda & -.81 \\ 1 & -\lambda \end{bmatrix} \Rightarrow (1.8-\lambda)(-\lambda) + .81 = \lambda^2 - 1.8\lambda + .81 = 0$

$(\lambda - .9)^2 = 0 \quad \lambda = .9$ Origin attracts along only eigenvector

$\lambda_1 = .9$ $\begin{bmatrix} .9 & -.81 \\ 1 & -.9 \end{bmatrix}$ $\begin{matrix} x_1 = .9x_2 \\ x_2 = x_2 \end{matrix}$ $\vec{v_1} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$\vec{X_0} = \begin{bmatrix} 15 \\ 3 \end{bmatrix}$ $\vec{X_1} = \begin{bmatrix} 24.57 \\ 15 \end{bmatrix}$, $\vec{X_2} = \begin{bmatrix} 32.08 \\ 24.57 \end{bmatrix}$ $\vec{X_3} = \begin{bmatrix} 37.84 \\ 32.08 \end{bmatrix}$ $\quad \vec{X_n} = c_1 \begin{bmatrix} 9 \\ 10 \end{bmatrix} (.9)^n + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{X_4} = \begin{bmatrix} 42.12 \\ 37.84 \end{bmatrix}$ $\vec{X_5} = \begin{bmatrix} 45.17 \\ 42.12 \end{bmatrix}$ $\vec{X_6} = \begin{bmatrix} 47.19 \\ 45.17 \end{bmatrix}$ $\vec{X_7} = \begin{bmatrix} 48.35 \\ 47.19 \end{bmatrix}$ $\vec{X_8} = \begin{bmatrix} 48.81 \\ 48.35 \end{bmatrix}$ $\vec{X_9} = \begin{bmatrix} 48.698 \\ 48.81 \end{bmatrix}$

$\vec{X_{10}} = \begin{bmatrix} 48.11 \\ 48.70 \end{bmatrix}$ $\vec{X_{11}} = \begin{bmatrix} 47.17 \\ 48.12 \end{bmatrix}$ $\vec{X_{12}} = \begin{bmatrix} 45.92 \\ 47.166 \end{bmatrix}$ $\vec{X_{13}} = \begin{bmatrix} 44.46 \\ 45.92 \end{bmatrix}$ $\vec{X_{14}} = \begin{bmatrix} 42.83 \\ 44.46 \end{bmatrix}$ $\vec{X_{15}} = \begin{bmatrix} 41.08 \\ 42.83 \end{bmatrix}$

7. $A = \begin{bmatrix} .978 & -.006 \\ .004 & .992 \end{bmatrix}$ $\quad (.978-\lambda)(.992-\lambda) + .000024 = \lambda^2 - 1.97 + .9702 = 0$

$\lambda = \dfrac{1.97 \pm \sqrt{3.8809 - .38808}}{2} = \dfrac{1.97 \pm .01}{2} \Rightarrow \lambda_1 = .99, \ \lambda_2 = .98$

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7 cont'd. Origin attracts

$\lambda_1 = .99$

$\begin{bmatrix} -.012 & -.006 \\ .004 & .002 \end{bmatrix}$ $\quad x_1 = \frac{1}{2}x_2, \; x_2 = x_2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = .98$

$\begin{bmatrix} -.002 & -.006 \\ .004 & .012 \end{bmatrix}$ $\quad x_1 = -3x_2, \; x_2 = x_2 \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

8.a. $\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix} \Rightarrow (1-\lambda)(-4-\lambda)+6 = \lambda^2+3\lambda-4+6 = \lambda^2+3\lambda+2=0$

$(\lambda+2)(\lambda+1)=0 \quad \lambda=-2,-1$

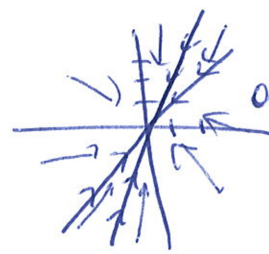
$\lambda_1=-2$

$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$ $\quad x_1 = \frac{2}{3}x_2, \; x_2 = x_2 \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda_2 = -1$

$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ $\quad x_1 = x_2, \; x_2 = x_2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = c_1 e^{-2t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



origin attracts

b. $\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} \Rightarrow (-2-\lambda)(-2-\lambda)-1 =$

$\lambda^2+4\lambda+4-1 = \lambda^2+4\lambda+3=0$

$(\lambda+3)(\lambda+1)=0 \quad \lambda=-3,-1$ $\quad$ Origin attracts

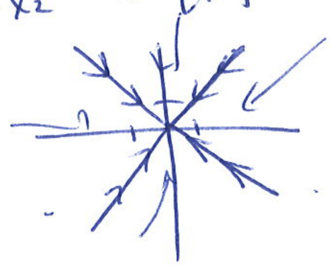
$\lambda_1=-3$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad x_1 = -x_2, \; x_2 = x_2 \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad x_1 = x_2, \; x_2 = x_2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t}$



c. $\begin{bmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{bmatrix} \Rightarrow (4-\lambda)(-6-\lambda)+24 = \lambda^2+2\lambda=0$

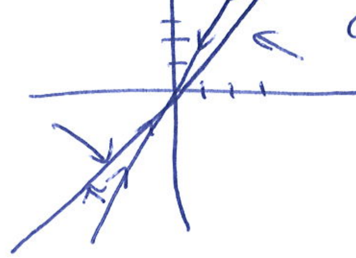
$\lambda(\lambda+2)=0 \quad \lambda=0,-2$

$\lambda_1=0$ $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$ $\quad x_1 = \frac{3}{4}x_2, \; x_2 = x_2 \quad \vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\lambda_2=-2$ $\begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix}$ $\quad x_1 = \frac{1}{2}x_2, \; x_2 = x_2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{x} = c_1\begin{bmatrix} 3 \\ 4 \end{bmatrix}e^{0} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-2t}$

$\lambda=0$ stable



attracts to stable line

202 Homework #10 key

6d. $\begin{bmatrix} 4-\lambda & -3 \\ 6 & -2-\lambda \end{bmatrix} \Rightarrow (4-\lambda)(-2-\lambda)+18 = \lambda^2-2\lambda-8+18 = \lambda^2-2\lambda+10=0$

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$\begin{bmatrix} 4-1-3i & -3 \\ 6 & -2-1-3i \end{bmatrix} = \begin{bmatrix} 3-3i & -3 \\ 6 & -3-3i \end{bmatrix} \div 3$ $\quad \cancel{2}x_1 = \frac{(1+i)}{2}x_2$, $x_2 = x_2$ $\quad \vec{v}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^t(\cos 3t + i\sin 3t) = e^t\begin{bmatrix} \cos 3t + i\sin 3t + i\cos 3t - \sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}$

$\vec{X} = c_1 e^t \begin{bmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix}$ Spirals outward

e. $\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix} \Rightarrow (3-\lambda)(-1-\lambda)+8 = \lambda^2-2\lambda-3+8 = \lambda^2-2\lambda+5=0$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\begin{bmatrix} 3-1-2i & -2 \\ 4 & -1-1-2i \end{bmatrix} = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \div 2$ $\quad \cancel{2}x_1 = \frac{(1+i)}{2}x_2$, $x_2 = x_2$ $\quad \vec{v}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^t(\cos 2t + i\sin 2t) = e^t\begin{bmatrix} \cos 2t + i\sin 2t + i\cos 2t - \sin 2t \\ 2\cos 2t + i2\sin 2t \end{bmatrix}$ Spirals outward

$\vec{X} = c_1 e^t \begin{bmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}$

7a. $13x^2 - 8xy + 7y^2 - 45 = 0$

$\begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix}$ $\quad (13-\lambda)(7-\lambda)-16 = \lambda^2 - 20\lambda + 75 = 0 \quad (\lambda-5)(\lambda-15)=0$

$\lambda_1 = 5, \lambda_2 = 15$

$15(x')^2 + 5(y')^2 - 45 = 0 \Rightarrow \frac{15(x')^2 + 5(y')^2 = 45}{45}$

$\frac{(x')^2}{3} + \frac{(y')^2}{9} = 1$ ellipse

b. $2x^2 - 4xy + 5y^2 - 36 = 0$ $\quad \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \Rightarrow (2-\lambda)(5-\lambda) - 4 = \lambda^2 - 7\lambda + 10 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0 \quad (\lambda-6)(\lambda-1) = 0$

$\lambda = 6, \lambda = 1$

7b cont'd

$(x')^2 + 6(y')^2 - 36 = 0 \Rightarrow \dfrac{(x')^2 + 6(y')^2 = 36}{36} \quad \dfrac{(x')^2}{36} + \dfrac{(y')^2}{6} = 1$

ellipse

c. $8x^2 + 8xy + 8y^2 + 10\sqrt{2}\,x + 26\sqrt{2}\,y + 31 = 0$

$\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \Rightarrow (8-\lambda)^2 - 16 = \lambda^2 - 16\lambda + 64 - 16 = \lambda^2 - 16\lambda + 48 = 0 \quad (\lambda-4)(\lambda-12)=0$

$\lambda = 4, \lambda = 12$

$4(x')^2 + 12(y')^2 + \begin{bmatrix} 10\sqrt{2} & 2\sqrt{2} \end{bmatrix} P X' + 31 = 0$ ellipse

P: $\lambda_1 = 4 \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 12 \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \div \sqrt{2} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} 10\sqrt{2} & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -10+2 & 10+2 \end{bmatrix}$

$= \begin{bmatrix} -8 & 12 \end{bmatrix}$

$4(x')^2 + 12(y')^2 - 8(x') + 12(y') + 31 = 0$

d. $xy + x - 2y + 3 = 0 \quad \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \quad (-\lambda)(-\lambda) - 1/4 = \lambda^2 - 1/4 = 0$

$\lambda = \pm 1/2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\frac{1}{2}(x')^2 - \frac{1}{2}(y')^2 + \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} X' + 3 = 0$

$(\frac{1}{2})(x')^2 - \frac{1}{2}(y')^2 - \frac{1}{\sqrt{2}}x' - \frac{3}{\sqrt{2}}y' + 3 = 0$ hyperbola

e. $3x^2 - 2xy + 3y^2 + 8z^2 - 16 = 0 \quad \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \Rightarrow (3-\lambda)^2 - 1 =$

$4(x')^2 + 2(y')^2 + 8z^2 - 16 = 0$ no change in z $\quad \lambda^2 - 6\lambda + 8 = 0 \quad (\lambda-4)(\lambda-2)=0$

$\lambda = 4, 2$

ellipsoid $\dfrac{(x')^2}{4} + \dfrac{(y')^2}{8} + \dfrac{z^2}{2} = 1$

f. $x^2 + 2y^2 + 2z^2 + 2yz - 1 = 0 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow (2-\lambda)^2 - 1 =$

$\lambda^2 - 4\lambda + 3 = 0$

$x^2 + 3(y')^2 + (z')^2 - 1 = 0$ no change in x $\quad (\lambda-3)(\lambda-1)=0$

$\lambda = 3, 1$

ellipsoid $x^2 + \dfrac{(y')^2}{(1/3)} + (z')^2 = 1$