MAT 202, Exam #3, Part I, Summer 2016 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (1 point each)

a. T **F** The length (norm) of a vector is given by $\|\vec{v}\| = |v_1 + v_2 + \cdots + v_n|$. $v_1^2+v_2^2+\cdots+v_n^2$

b. T **F** If $\vec{u}\cdot\vec{v} < 0$, then the angle θ between $\vec{u}$ and $\vec{v}$ is acute. obtuse

c. T **F** The dot product is the only inner product that can be defined on R^n.

d. **T** F A set of vectors in an inner product space V is orthogonal when every pair of vectors in S is orthogonal.

e. **T** F If a set of nonzero vectors S in an inner product space V is orthogonal, then S is linearly independent.

f. T **F** The orthogonal complement of R^n is the empty set. $\{\vec{0}\}$

g. T **F** The function $f(x) = \cos x$ is a linear transformation of $R \to R$.

h. T **F** The set of all vectors mapped from a vector space V into another vector space W by a linear transformation T is the kernel of T. kernel maps to $\vec{0}$

i. **T** F The vector spaces R^3 and $M_{3,1}$ are isomorphic to each other.

j. **T** F A linear transformation $T: V \to W$ is one-to-one when the preimage of every $\vec{w}$ in the range consists of a single vector $\vec{v}$.

k. T **F** All linear transformations T have a unique inverse T^{-1}. (not if not one-to-one)

l. **T** F In general, the compositions $T_2 \circ T_1$ and $T_1 \circ T_2$ have the same standard matrix A.

m. T **F** Two matrices that represent the same linear transformation $T: V \to V$ with respect to a different basis are not necessarily similar. they are similar

n. T **F** The matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is one-to-one. it is onto, but not one-to-one

o. **T** F The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ scales a vector vertically.

2. Given the vector $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -5 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$, find the following: (2 points each)

a. $||\vec{u}||$

$$\sqrt{4+25+25} = \sqrt{54} = 3\sqrt{6}$$

b. A unit vector in the direction of $\vec{u}$

$$\hat{u} = \begin{bmatrix} \frac{2}{3\sqrt{6}} \\ 0 \\ \frac{-5}{3\sqrt{6}} \\ \frac{5}{3\sqrt{6}} \end{bmatrix}$$

c. $||\vec{u} + \vec{v}||$

$$\vec{u}+\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -6 \\ 6 \end{bmatrix}$$

$$||\vec{u}+\vec{v}|| = \sqrt{4+4+36+36} = \sqrt{80} = 4\sqrt{5}$$

d. $\vec{u} \cdot \vec{v}$

$$= 0+0+5+5 = 10$$

e. Are $\vec{u}$ and $\vec{v}$ orthogonal? If not, is the angle between the vectors acute or obtuse?

No. acute since $\vec{u} \cdot \vec{v} > 0$

3. Explain why $\vec{u} + (\vec{u} \cdot \vec{v})$ is meaningless. (3 points)

$\vec{u} \cdot \vec{v}$ is a # and cannot be added to a vector

4. Given the inner product $\langle f, g \rangle = \int_{-2}^{2} f(x)g(x)dx$, find a second-degree polynomial $ax^2 + bx + c$ orthogonal to $x + 1$. (5 points)

$$\int_{-2}^{2}(x+1)(ax^2+bx+c)dx = \int_{-2}^{2} ax^3+bx^2+cx+ax^2+bx+c\,dx =$$

$$\underbrace{\int_{-2}^{2} ax^3+cx+bx\,dx}_{\text{odd}}{}^{\to 0} + \int_{-2}^{2} bx^2+ax^2+c\,dx = 2\int_0^2 \underbrace{bx^2+ax^2+c}_{\text{even}}\,dx =$$

$$2\left[\frac{b}{3}x^3 + \frac{a}{3}x^3 + cx\right]_0^2 = 2\left[\frac{8b}{3} + \frac{8a}{3} + 2c\right] = 0$$

$$8b+8a+6c=0 \Rightarrow 4b+4a=-3c$$

let $a=b=3 \Rightarrow 12+12=-3c \Rightarrow 24=-3c \Rightarrow c=-8$

$g(x) = 3x^2+3x-8$ is orthogonal to $x+1$. answers will vary.

5. Determine if $T: R^2 \to R^2$, $T(x,y) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ is a linear transformation. Prove it if it is; find a counterexample if it is not, and state the property that is violated. (5 points)

It is not. fails addition and scalar multiplication

$T(x,y) + T(a,b) = \begin{bmatrix} x^2+a^2 \\ y+b \end{bmatrix}$ but $T(x+a, y+b) = \begin{bmatrix} (x+a)^2 \\ y+b \end{bmatrix}$ $(x+a)^2 \neq x^2+a^2$

or $-4T(x,y) = \begin{bmatrix} -4x^2 \\ -4y \end{bmatrix}$, but $T(-4x,-4y) = \begin{bmatrix} 16x^2 \\ -4y \end{bmatrix}$ $-4x^2 \neq 16x^2$

6. Use Gram-Schmidt to find an orthogonal basis for the space spanned by $\left\{ \underset{\vec{u}_1}{\begin{bmatrix}0\\1\\1\end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix}1\\-1\\1\end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix}1\\1\\1\end{bmatrix}} \right\}$. (7 points)

$$\vec{v}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix} = \vec{u}_1$$

$$\vec{v}_2 : \quad \frac{\vec{v}_1 \cdot \vec{u}_2}{\|\vec{v}_1\|^2}\vec{v}_1 \qquad \vec{v}_2 = \vec{u}_2 - \text{proj}_{\vec{v}_1}\vec{u}_2$$

$$\frac{0-1+1}{(\sqrt{2})^2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \frac{0}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\vec{v}_3 : \quad \frac{\vec{v}_1 \cdot \vec{u}_3}{\|\vec{v}_1\|^2}\vec{v}_1 \qquad \frac{\vec{v}_2 \cdot \vec{u}_3}{\|\vec{v}_2\|^2}\vec{v}_2 \qquad \vec{v}_3 = \vec{u}_3 - \text{proj}_{\vec{v}_1}\vec{u}_3 - \text{proj}_{\vec{v}_2}\vec{u}_3$$

$$\frac{0+1+1}{(\sqrt{2})^2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \frac{2}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} \qquad \frac{1-1+1}{(\sqrt{3})^2}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1/3\\-1/3\\1/3\end{bmatrix} = \begin{bmatrix}2/3\\1/3\\-1/3\end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix}2\\1\\-1\end{bmatrix}$$

$$\left\{ \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix} \right\}$$

7. For $W = span\left\{ \overset{\vec{w}_1}{\begin{bmatrix}1\\2\\0\\1\end{bmatrix}}, \overset{\vec{w}_2}{\begin{bmatrix}0\\1\\0\\-2\end{bmatrix}} \right\}$, find the projection of $\vec{y} = \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$ onto W, and its orthogonal complement in $W^{\perp}$. (6 points)

$$\vec{y}_{\parallel} = \frac{\vec{y}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 + \frac{\vec{y}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\vec{w}_2 = \frac{0+2+0+1}{(\sqrt{1+4+1})^2}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + \frac{0+1+0-2}{(\sqrt{1+4})^2}\begin{bmatrix}0\\1\\0\\-2\end{bmatrix} =$$

$$\frac{3}{6}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + \frac{-1}{5}\begin{bmatrix}0\\1\\0\\-2\end{bmatrix} = \begin{bmatrix}1/2\\1\\0\\1/2\end{bmatrix} + \begin{bmatrix}0\\-1/5\\0\\2/5\end{bmatrix} = \begin{bmatrix}1/2\\4/5\\0\\9/10\end{bmatrix} \text{ in } W$$

$$\vec{y}_{\perp} = \begin{bmatrix}0\\1\\-1\\1\end{bmatrix} - \begin{bmatrix}1/2\\4/5\\0\\9/10\end{bmatrix} = \begin{bmatrix}-1/2\\1/5\\-1\\1/10\end{bmatrix} \text{ in } W^{\perp}.$$

8. Write the matrix for the linear transformation for the derivative on the basis $\{e^{2x}, xe^{2x}, x^2e^{2x}\}$. Then apply that transformation to the vector $y(x) = 5e^{2x} + 3xe^{2x} + x^2e^{2x}$. What is y' in this basis? Use your transformation matrix to find it. (5 points)

$\frac{d}{dx}e^{2x} = 2e^{2x}$ $\quad \frac{d}{dx}xe^{2x} = e^{2x} + 2xe^{2x}$ $\quad \frac{d}{dx}x^2e^{2x} = 2xe^{2x} + 2x^2e^{2x}$

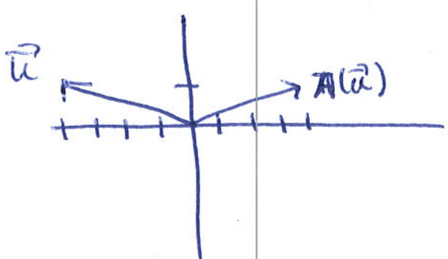
$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+3+0 \\ 0+6+2 \\ 0+0+2 \end{bmatrix} = \begin{bmatrix} 13 \\ 8 \\ 2 \end{bmatrix}$$

$$13e^{2x} + 8xe^{2x} + x^2e^{2x} = y'$$

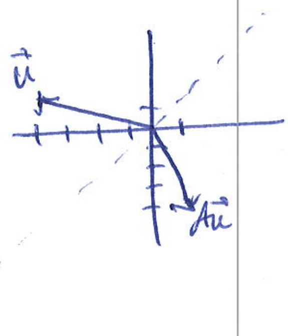
9. Given the vector $\vec{u} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and the transformations described below, write the matrix of the transformation and apply it to $\vec{u}$ and graph both $\vec{u}$ and $A\vec{u}$. (2 points each)

a. Reflection over the y-axis



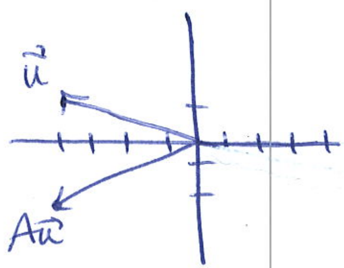
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A\vec{u} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

b. Reflection over the line $y = x$



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A\vec{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

c. Vertical stretch by a factor of -2



$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A\vec{u} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

MAT 202, Exam #3, Part II, Summer 2016 Name ______________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Given the linear transformation defined by $A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$, find a basis for the kernel and range of the transformation. (6 points)

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_3 - x_5 &= 0 \\ x_2 - x_3 - 2x_5 &= 0 \\ x_3 &= x_3 \\ x_4 + 4x_5 &= 0 \\ x_5 &= x_5 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -2x_3 + x_5 \\ x_2 &= x_3 + 2x_5 \\ x_3 &= x_3 \\ x_4 &= -4x_5 \\ x_5 &= x_5 \end{aligned}$$

$$\text{Ker}(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

$$\text{rang}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

2. Given the linear transformation defined by $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{bmatrix}$, determine if the transformation is any of the following. Explain your reasoning in each case. (2 points each)

a. One-to-one

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no (not a pivot in every column)

b. Onto

no (not a pivot in every row)

c. An isomorphism from R^4 to R^4.

no. it must be both one-to-one & onto but it is neither

3. Given $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_3 \\ x_1 + 2x_2 - x_4 \\ x_4 \end{bmatrix}$. Write the matrix of the transformation. Explain why this proves the transformation is linear. If T^{-1} exists, find its matrix. (6 points)

$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. matrices obey the same properties a linear transformations: ie. $A\vec{0} = \vec{0}$, $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$, $kA\vec{u} = A(k\vec{u})$.

$A^{-1} = T^{-1} = \begin{bmatrix} 2/3 & 0 & 1/3 & 1/3 \\ -1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ transformation is one-to-one and onto.

4. Prove that if $A\vec{x} = \lambda\vec{x}$, and if A is similar to B, then there exists a P such that $P^{-1}BP\vec{x} = \lambda\vec{x}$. (4 points)

Suppose $A\vec{x} = \lambda x$. Since A is similar to be, by definition (of similarity). the must exist a P such that $A = P^{-1}BP$.

Substituting implies $P^{-1}BP\vec{x} = \lambda\vec{x}$.

5. Consider $S = span\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$. Find an orthogonal basis for $S^{\perp}$. (5 points)

in $S^{\perp} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ dot w/ vectors in S

$\left.\begin{matrix}a+2b+c=0\\ d=0\end{matrix}\right\}$ conditions on vectors in $S^{\perp}$ $\Rightarrow$ $a=-2b-c$

if $b=0$ $a=-c$ $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$

$S^{\perp} = span\left\{\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}\right\}$

new condition

$a+2b+c=0$
$d=0$
$a \quad -c=0$

$a=+c$
$b=-c$

$\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$

6. Find the QR factorization for $A = \begin{bmatrix}1&0\\0&0\\1&1\\1&2\end{bmatrix}$. (5 points)

$$\begin{bmatrix}0\\0\\1\\2\end{bmatrix} - \frac{0+0+1+2}{(\sqrt{3})^2}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

$Q = \begin{bmatrix}1&-1\\0&0\\1&0\\1&1\end{bmatrix}$

$$Q^TA = \begin{bmatrix}1&0&1&1\\-1&0&0&1\end{bmatrix}\begin{bmatrix}1&-1\\0&0\\1&1\\1&2\end{bmatrix} = \begin{bmatrix}1+0+1+1 & -1+0+1+2\\ -1+0+0+0 & 1+0+0+2\end{bmatrix}$$

$$= \begin{bmatrix}3&2\\0&3\end{bmatrix}$$

$Q = \begin{bmatrix}1&-1\\0&0\\1&0\\1&1\end{bmatrix}$ $R = \begin{bmatrix}3&2\\0&3\end{bmatrix}$

7. Consider a generic 5×7 matrix. Is it possible for the linear transformation defined by the matrix to be:

   a. One-to-one? (2 points)

   no – can't have 7 pivots (in columns) since there are only 5 rows

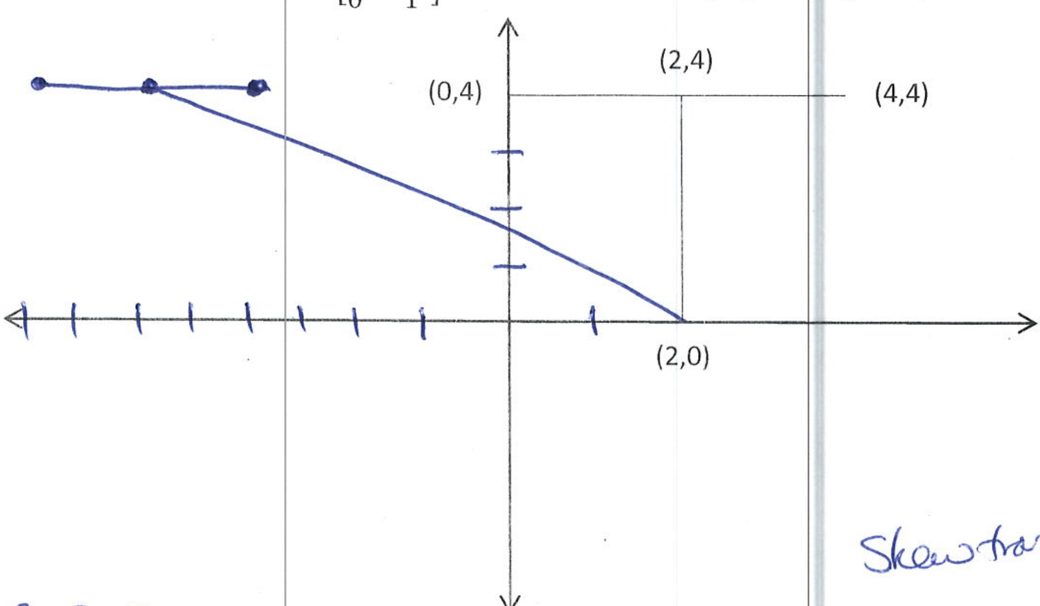
   b. Onto? (2 points)

   yes. There can be a pivot in every row

   c. If the matrix has 4 pivots, what is the dimension of the kernel and the range? (4 points)

   dim(ker(A)) = 3
   dim(range(A)) = 4

8. Consider the shape T defined by the vertices shown on the graph. Apply the shear transformation $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ and draw the resulting figure. (4 points)



Skew transformation

$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

9. Provide four examples of vector spaces that are isomorphic to R^4. (4 points)

$M_{2,2}$

P_3

$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$M_{4,1}$

answers will vary