Math 2568, Exam #2, Part I, Summer 2015 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. For each of the questions, assume that A is $n \times n$. (2 points each)

a. T **F** A system that does have a unique solution cannot be solved with Cramer's rule.

b. T **F** A matrix is invertible if the determinant of the matrix is 0.

c. **T** F If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

d. **T** F If zero is not an eigenvalue of A, then the determinant of A is non-zero.

e. **T** F If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of e^A.

f. T **F** If A is invertible, then A is diagonalizable with real numbers.

g. T **F** Row operations on a matrix do not change its eigenvalues.

h. T **F** Similar matrices always have the same eigenvectors.

i. T **F** The dimension of the eigenspace of an $n \times n$ matrix is always n.

j. **T** F The vector $\begin{bmatrix}1\\1\end{bmatrix}$ is not an eigenvector of $\begin{bmatrix}5 & -2\\7 & 8\end{bmatrix}$. $\begin{bmatrix}5 & -2\\7 & 8\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5-2\\7+8\end{bmatrix} = \begin{bmatrix}3\\15\end{bmatrix}$

k. T **F** A 7x7 matrix has four eigenvalues, two eigenspaces are one dimensional, two eigenspaces are two dimensional; therefore this matrix is diagonalizable. (this is only 6 dimensions)

l. **T** F Stochastic matrices, regardless of their size, always have at least one real eigenvector.

m. T **F** The real eigenvalues of a system of linear ODEs must always both attract or repel from the origin.

n. **T** F In a discrete dynamical system, the magnitude of λ determines whether a complex eigenvalues causes the origin to repel or attract.

o. T **F** The cross product is one type of inner product.

p. **T** F Normalizing a vector refers to making a vector pointing in a particular direction have components that satisfy certain conditions.

2. Find the determinant of the matrix $\begin{bmatrix}2&3&1&2&0\\5&0&-4&1&0\\4&-3&3&-1&2\\3&0&0&1&0\\4&2&5&2&0\end{bmatrix}$ by the cofactor method. (12 points)

$$2\begin{vmatrix}2&3&1&2\\5&0&-4&1\\3&0&0&1\\4&2&5&2\end{vmatrix} = 2\left[3\begin{vmatrix}3&1&2\\0&-4&1\\2&5&2\end{vmatrix} - 1\begin{vmatrix}2&3&1\\5&0&4\\4&2&5\end{vmatrix}\right]$$

$$\begin{vmatrix}3&1&2\\0&-4&1\\2&5&2\end{vmatrix} = 3\begin{vmatrix}-4&1\\5&2\end{vmatrix} + 2\begin{vmatrix}1&2\\-4&1\end{vmatrix} = 3(-8-5)+2(1+8) = 3(-13)+2(9) = -39+18 = -21$$

$$\begin{vmatrix}2&3&1\\5&0&-4\\4&2&5\end{vmatrix} = -5\begin{vmatrix}3&1\\2&5\end{vmatrix} - (-4)\begin{vmatrix}2&3\\4&2\end{vmatrix} = -5\left((15-2)+4(4-12\right) = -5(13)+4(-8) = -65-32 = -97$$

$$2\left[3(-21)-(-97)\right] = 2\left[-63+97\right] = 2(34) = 68$$

3. Find the determinant of the matrix $\begin{bmatrix}1&-2&5\\2&3&-2\\4&0&1\end{bmatrix}$ by the row-reducing method. (11 points)

$-2R_1+R_2 \to R_2$
$-4R_1+R_3 \to R_3$

$$\begin{bmatrix}1&-2&5\\0&7&-12\\0&8&-19\end{bmatrix}$$

no change

$\frac{1}{7}R_2 \to R_2$

$$\begin{bmatrix}1&-2&5\\0&1&-12/7\\0&8&-19\end{bmatrix}$$

Change by 1/7 $\quad$ *7 to correct

$-8R_2+R_3 \to R_3$

$$\begin{bmatrix}1&-2&5\\0&1&-12/7\\0&0&-37/7\end{bmatrix}$$

$\Rightarrow \det = -37/7$

$-19 + 96/7 = \frac{-133+96}{7} =$

original $= -37/7 * 7 = -37$

$$\begin{array}{r}19\\7\\\hline 133\end{array}$$

4. Given that A and B are 3 × 3 matrices with det A = -2 and det B = 5, find the following. (4 points each)

a) det AB

-10

d) det B^T

5

b) det A^{-1}

$-\frac{1}{2}$

e) det 3A

$3^3(-2) = -54$

c) det $(-AB^5)$

$(-1)^3(-2)5^5 = 6250$

5. For each of the matrices shown below, find the eigenvalues and eigenvectors of the matrix. (7 points each)

a. $A = \begin{bmatrix} 6 & -6 \\ -6 & -3 \end{bmatrix}$

$(6-\lambda)(-3-\lambda) - 36 = 0$

$\lambda^2 - 3\lambda - 18 - 36 = 0$

$\lambda^2 - 3\lambda - 54 = 0$

$(\lambda - 9)(\lambda + 6) = 0$

$\lambda = 9, \lambda = -6$

$\lambda_1 = 9$ $\begin{bmatrix} 6-9 & -6 \\ -6 & -3-9 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -6 & -12 \end{bmatrix}$

$\Rightarrow 3x_1 + 6x_2 = 0$

$x_1 = -2x_2$
$x_2 = x_2$ $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = -6$ $\begin{bmatrix} 6-(-6) & -6 \\ -6 & -3-(-6) \end{bmatrix} = \begin{bmatrix} 12 & -6 \\ -6 & 3 \end{bmatrix}$

$12x_1 - 6x_2 = 0$

$x_1 = \frac{1}{2}x_2$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$x_2 = x_2$

b. $B = \begin{bmatrix} -2 & -3 \\ -1 & 0 \end{bmatrix}$

$(-2-\lambda)(-\lambda) - 3 = \lambda^2 + 2\lambda - 3 = 0$

$(\lambda + 3)(\lambda - 1) = 0$ $\lambda = -3, 1$

$\lambda_1 = -3$

$\begin{bmatrix} -2-(-3) & -3 \\ -1 & -(-3) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}$

$x_1 - 3x_2 = 0$

$x_1 = 3x_2$ $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
$x_2 = x_2$

$\lambda_2 = 1$ $\begin{bmatrix} -2-1 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix}$

$-x_1 - x_2 = 0$

$x_1 = -x_2 = 0$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$x_2 = x_2$

6. Find the equilibrium vector of the matrix $P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ algebraically. Be sure to properly normalize the vector. (8 points)

$P - I$

$\begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}$

$-.2x_1 + .3x_2 = 0$

$x_1 = \frac{3}{2}x_2$

$x_2 = x_2$

$\vec{x} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$3 + 2 = 5$

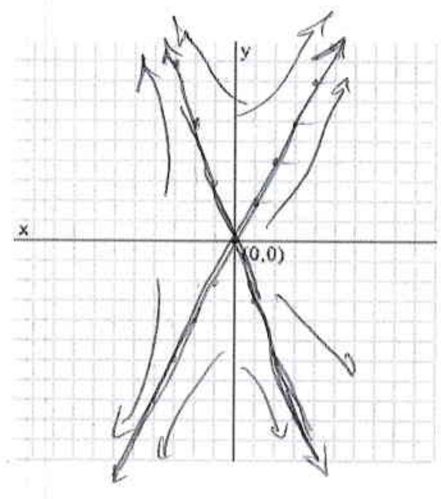
$\vec{q} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$

7. For each of the situations below, determine the properties of the linear system of ODEs. Is the origin an attractor, a repeller, or a saddle point? Sketch the eigenvalues on the graphs provided (if they are real) and plot some sample trajectories. (5 points each)

   a. $\lambda_1 = 0.97, \lambda_2 = 0.2, \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

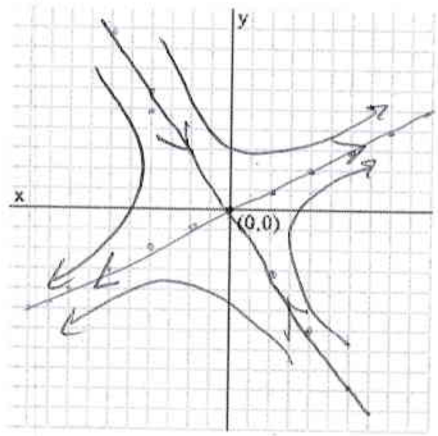
   both pos.

   origin repels



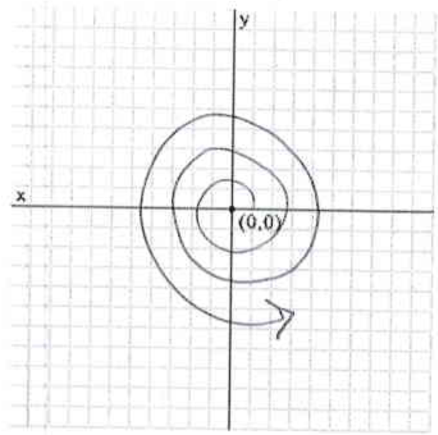
b. $\lambda_1 = \frac{7}{5}, \lambda_2 = -2, \vec{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-2\\3\end{bmatrix}$.

Saddle point



c. $\lambda_1 = \frac{1}{2} + \frac{7}{2}i, \lambda_2 = \frac{1}{2} - \frac{7}{2}i$.

real part positive
origin repels



Spirals out

8. For the vectors $\vec{u} = \begin{bmatrix}1\\4\\-4\end{bmatrix}, \vec{v} = \begin{bmatrix}4\\-2\\-1\end{bmatrix}$, find the following: (4 points each)

a. $||\vec{v}||$

$\sqrt{16+4+1} = \sqrt{21}$

b. A unit vector in the direction of $\vec{u}$.

$\|u\| = \sqrt{1+16+16} = \sqrt{33}$

$\hat{u} = \begin{bmatrix} 1/\sqrt{33} \\ 4/\sqrt{33} \\ -4/\sqrt{33} \end{bmatrix}$

c. $\vec{u} \cdot \vec{v}$

$4 - 8 + 4 = 0$

d. Are the two vectors orthogonal? Why or why not?

they are since the dot product is zero

Math 2568, Exam #2, Part II, Summer 2015 Name ______________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Use Cramer's rule to find the solution to the system $\begin{cases} x_1 + 3x_2 - 2x_3 + x_4 = 12 \\ 2x_1 - x_2 + x_3 + 4x_4 = 2 \\ \quad 2x_2 - 3x_3 + 2x_4 = 12 \\ 3x_1 \quad + 2x_3 - 5x_4 = -6 \end{cases}$. Write all the required matrices and their determinants, but you may calculate the determinants with your calculator. (16 points)

$$\det A = \begin{vmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 1 & 4 \\ 0 & 2 & -3 & 2 \\ 3 & 0 & 2 & -5 \end{vmatrix} = -88$$

$$\det A_1 = \begin{vmatrix} 12 & 3 & -2 & 1 \\ 2 & -1 & 1 & 4 \\ 12 & 2 & -3 & 2 \\ -6 & 0 & 2 & -5 \end{vmatrix} = -88$$

$$\det A_2 = \begin{vmatrix} 1 & 12 & -2 & 1 \\ 2 & 2 & 1 & 4 \\ 0 & 12 & -3 & 2 \\ 3 & -6 & 2 & -5 \end{vmatrix} = -176$$

$$\det A_3 = \begin{vmatrix} 1 & 3 & 12 & 1 \\ 2 & -1 & 2 & 4 \\ 0 & 2 & 12 & 2 \\ 3 & 0 & -6 & 5 \end{vmatrix} = 176$$

$$\det A_4 = \begin{vmatrix} 1 & 3 & -2 & 12 \\ 2 & -1 & 1 & 2 \\ 0 & 2 & -3 & 12 \\ 3 & 0 & 2 & -6 \end{vmatrix} = -88$$

$$x_1 = \frac{\det A_1}{\det A} = \frac{-88}{-88} = 1$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{-176}{-88} = 2$$

$$x_3 = \frac{\det A_3}{\det A} = \frac{176}{-88} = -2$$

$$x_4 = \frac{\det A_4}{\det A} = \frac{-88}{-88} = 1$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

2. Find the similarity transformation for the matrix $B = \begin{bmatrix} 9 & -3 \\ -6 & 1 \end{bmatrix}$ that converts this matrix into a similar rotation matrix. Then use that matrix to find the angle of rotation. Give your angle in radians rounded to 4 decimal places, or in degrees rounded to one decimal place. (12 points)

$(9-\lambda)(1-\lambda) - 18 = \lambda^2 - 10\lambda + 9 - 18 = \lambda^2 - 10\lambda - 9 = 0$

$\lambda = \frac{10 \pm \sqrt{100+36}}{2} = \frac{10 \pm 2\sqrt{34}}{2} = 5 \pm \sqrt{34}$

Cannot be similar to a rotation matrix since eigenvalues are real

3. Solve the discrete dynamical system given by $\vec{x}_{k+1} = \begin{bmatrix} 0.3 & 0.4 \\ -0.5 & 1.1 \end{bmatrix} \vec{x}_k$. Find the eigenvalues and eigenvectors. Write the solution in the form $\vec{x}_n = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2$. (12 points)

$(.3-\lambda)(1.1-\lambda) + (.5)(.4) = 0$

$\lambda^2 - 1.4\lambda + .33 + .2 = 0$

$\lambda^2 - 1.4\lambda + .53 = 0$

$\lambda = \frac{1.4 \pm \sqrt{(1.4)^2 - 4(.53)}}{2} = \frac{1.4 \pm \sqrt{-.16}}{2} = \frac{1.4 \pm .4i}{2} = .7 \pm .2i$

$\begin{bmatrix} .3-(.7+.2i) & .4 \\ -.5 & 1.1-(.7+.2i) \end{bmatrix} = \begin{bmatrix} -.4-.2i & .4 \\ -.5 & -.4-.2i \end{bmatrix}$

$.7^2 + .2^2 < 1$

$.5x_1 + (.4+.2i)x_2 = 0$

$x_1 = -\frac{(.4+.2i)}{.5}x_2 = (.8+.4i)x_2$

$x_2 = x_2$

$\vec{v}_1 = \begin{bmatrix} .8+.4i \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} .8-.4i \\ 1 \end{bmatrix}$

Origin spirals in

$\vec{x}_n = c_1(.7 \pm .2i)^n \begin{bmatrix} .8+.4i \\ 1 \end{bmatrix} + c_2(.7-.2i)^n \begin{bmatrix} .8-.4i \\ 1 \end{bmatrix}$

4. Solve the system of ODEs given by $\vec{x}' = \begin{bmatrix} 2 & 9 \\ 1 & 2 \end{bmatrix}\vec{x}$. Sketch a graph of the eigenvectors and plot some sample trajectories. Is the origin an attractor, a repeller or a saddle point? Give your final solution in exact form with e. (12 points)

$(2-\lambda)(2-\lambda)-9=0$

$\lambda^2-4\lambda+4-9=0$

$\lambda^2-4\lambda-5=0$

$(\lambda-5)(\lambda+1)=0$

$\lambda=5,\ \lambda=-1$

origin is a saddle point

$\vec{X}=c_1e^{5t}\begin{bmatrix}3\\1\end{bmatrix}+c_2e^{-t}\begin{bmatrix}-3\\1\end{bmatrix}$

$\lambda_1=5 \quad \begin{bmatrix}2-5 & 9\\ 1 & 2-5\end{bmatrix}=\begin{bmatrix}-3 & 9\\ 1 & -3\end{bmatrix} \Rightarrow \begin{matrix} x_1-3x_2=0 \\ x_1=3x_2 \\ x_2=x_2\end{matrix} \quad \vec{v}_1=\begin{bmatrix}3\\1\end{bmatrix}$

$\lambda_1=-1 \quad \begin{bmatrix}2-(-1) & 9\\ 1 & 2-(-1)\end{bmatrix}=\begin{bmatrix}3 & 9\\ 1 & 3\end{bmatrix} \Rightarrow \begin{matrix} x_1+3x_2=0 \\ x_1=-3x_2 \\ x_2=x_2\end{matrix} \quad \vec{v}_2=\begin{bmatrix}-3\\1\end{bmatrix}$

5. The vectors $\vec{u}=\begin{bmatrix}2\\-3\\1\\0\end{bmatrix}, \vec{v}=\begin{bmatrix}3\\2\\0\\1\end{bmatrix}$ are orthogonal to each other. Find a vector $\vec{w}=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ orthogonal to both. (10 points)

$\begin{bmatrix}2 & -3 & 1 & 0\\ 3 & 2 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 2/13 & 3/13\\ 0 & 1 & -3/13 & 2/13\end{bmatrix}$

$x_1+2/13\,x_3+3/13\,x_4=0$

$x_2-3/13\,x_3+2/13\,x_4=0$

$\Rightarrow$

$x_1 = -2/13\,x_3 - 3/13\,x_4$

$x_2 = 3/13\,x_3 - 2/13\,x_4$

$x_3 = x_3$

$x_4 = x_4$

$\begin{bmatrix}-2\\3\\13\\0\end{bmatrix}, \begin{bmatrix}-3\\-2\\0\\13\end{bmatrix}$

6. Determine if the polynomials $p(t) = 1 - t, q(t) = t^2 + 3t$ are orthogonal under the inner product $\langle f|g\rangle \int_{-1}^{1} f(t)g(t)dt$. (8 points)

$$(1-t)(t^2+3t) = t^2+3t-t^3-3t^2$$

$$-t^3-2t^2+3t$$

$$\int_{-1}^{1} -t^3-2t^2+3t\,dt = \cancel{\int_{-1}^{1} -t^3+3t\,dt}^{\,0} + 2\int_{0}^{1} -2t^2\,dt = -4\left(\frac{t^3}{3}\right)\Big|_0^1 = \boxed{-\frac{4}{3}}$$

$\therefore$ not orthogonal