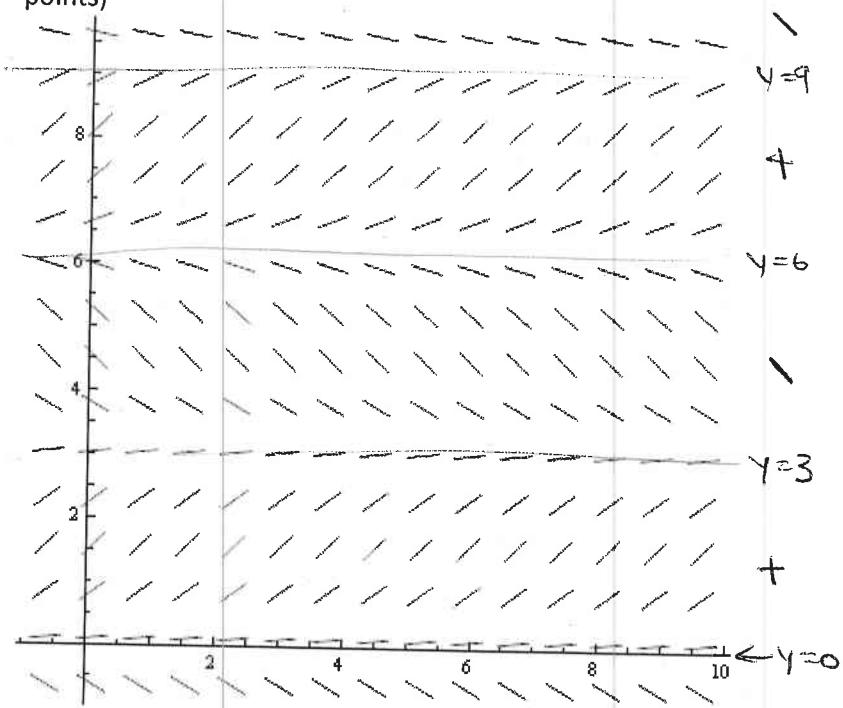
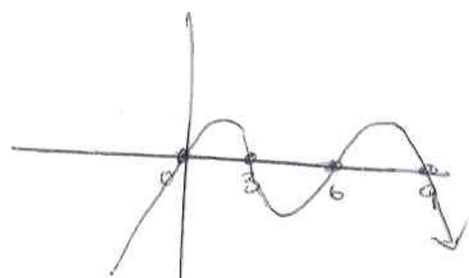
Math 2415, Exam #1, Summer 2015 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Assuming the equilibrium solutions are integers, use the graph below to sketch the phase portrait of the differential equation that produced the slope field shown here, and write the differential equation that produced it. You may assume the equilibrium values are integers. (7 points)





$$\frac{dy}{dx} = -y(y-3)(y-6)(y-9)$$

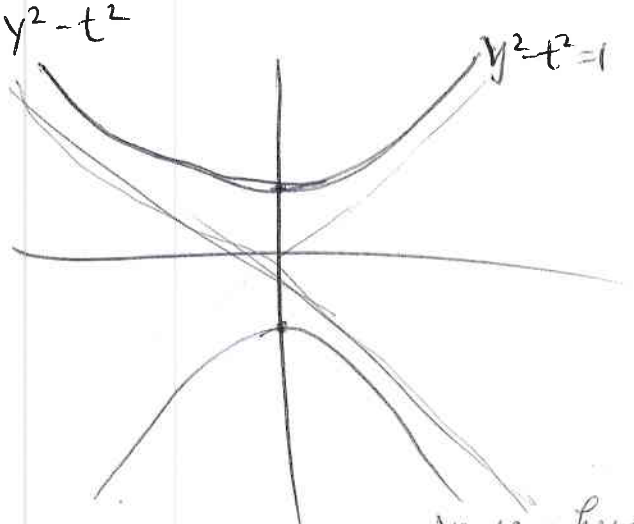
2. For the nonlinear differential equation $y' = \frac{y+2t}{(1-y^2+t^2)}$. Determine the region where the solution is not defined, and then sketch it in the plane. (6 points)

$$1 - y^2 + t^2 = 0 \Rightarrow 1 = y^2 - t^2$$

$$f(y,t) = \frac{y+2t}{1-y^2+t^2}$$

$$\frac{\partial f(y,t)}{\partial y} = \frac{1(1-y^2+t^2) - (y+2t)(-2y)}{(1-y^2+t^2)^2}$$

Same domain as $f(y,t)$



$y^2 - t^2 = 1$ hyperbola not defined

3. A tank has pure water flowing into it at 12 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 12 L/min. Initially, the tank contains 20 kg of salt in 5000 L of water. Find an equation to model the amount of salt in the tank at any time t. How much salt will there be in the tank after 1 hour? (12 points)

$$\frac{dQ}{dt} = \frac{12L}{min} \cdot \frac{0\,kg}{L} - \frac{12L}{min} \cdot \frac{Q\,kg}{5000L} \qquad Q(0) = 20$$

$$\frac{dQ}{dt} = -\frac{3Q}{1250}$$

$$\int \frac{dQ}{Q} = \int \frac{-3}{1250} dt$$

$$\ln Q = \frac{-3}{1250}t + C$$

$$Q = Q_0 e^{-3/1250 t}$$

$$\Rightarrow Q(t) = 20 e^{\left(\frac{-3}{1250}\right)t} \qquad 1 \text{ hour} = 60 \text{ min}$$

$$Q(60) = 20 e^{\frac{-3}{1250} \cdot 60} = 17.32 \text{ kg}$$

1. Solve the differential equation $y' = \frac{xy}{x^2 - y^2}$. (10 points) homogeneous

$$y = vx$$

$$y' = v'x + v$$

$$v'x + v = \frac{x \cdot vx}{x^2 - v^2x^2}$$

$$v'x + v = \frac{\cancel{x^2}\, v}{\cancel{x^2}(1-v^2)}$$

$$v'x = \frac{v}{1-v^2} - v\frac{(1-v^2)}{1-v^2}$$

$$v'x = \frac{v - v + v^3}{1-v^2} = \frac{v^3}{1-v^2}$$

$$\frac{dv}{dx} \cdot x = \frac{v^3}{1-v^2}$$

$$\frac{1-v^2}{v^3} dv = \frac{1}{x} dx$$

$$\int v^{-3} - v^{-1} dv = \int \frac{1}{x} dx$$

$$-\frac{1}{2}v^{-2} - \ln v = \ln x + C$$

$$-\frac{1}{2v^2} - \ln v = \ln x + C \qquad v = y/x$$

$$-\frac{1}{2\frac{y^2}{x^2}} - \ln(y/x) = \ln x + C$$

$$-\frac{x^2}{2y^2} + \ln(x/y) = \ln x + C$$

$$\boxed{\ln(x/y) - \frac{x^2}{2y^2} = \ln x + C}$$

4. Solve the differential equation $y' = \frac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$. (10 points)

$$\int y^{-3}dy = \int \frac{x}{\sqrt{1+x^2}}dx \qquad \begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}$$

$$-\frac{1}{2}y^{-2} = \int \frac{1}{2}u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} = \sqrt{1+x^2} + C$$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} + C$$

$$\frac{-1}{2(-1)^2} = \sqrt{1+0^2} + C$$

$$-\frac{1}{2} = 1 + C$$

$$-\frac{3}{2} = C$$

$$\boxed{-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}}$$

5. Use the method of integrating factors to solve the differential equation $ty' - 2y = t^3 sint$, $y(0) = 3$. (10 points)

$$y' - \frac{2}{t}y = t^2 \sin t$$

$$\mu = e^{\int -2/t\,dt} = e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2}$$

$$t^{-2}y' - 2t^{-3}y = \sin t$$

$$\int (t^{-2}y)' = \int \sin t$$

$$t^{-2}y = -\cos t + C$$

$$y = -t^2 \cos t + Ct^2$$

$$3 = -(0)\cos(0) + C(0)\ !$$

function not defined at $t=0$

6. Convert the Bernoulli equation $y' + 4ty = y^{-3}$ into a first order linear differential equation. You do not have to solve. (7 points)

$y^{-n} = y^{3}$ $\quad (1-n)y^{-n} = 4y^3$

$4y^3 y' + 16t y^4 = 4$

$z = y^4$

$z' = \frac{dz}{dt} = 4y^3 y'$

$\frac{dz}{dt} + 16tz = 4$

$z' + 16tz = 4$

$p(t) = 16t$

$g(t) = 1$

7. Use Euler's method to find the first three of 10 steps from $y(0) = 3$ to $y(1)$, under the differential equation $y' = \frac{y+2t}{(1-y^2+t^2)}$. (10 points)

$\Delta t = \frac{1-0}{10} = .1$

$y_0 = 3$ $\quad m = \frac{3+0}{1-9+0} = \frac{3}{-8}$ $\quad y_1 = \frac{3}{-8}(.1) + 3 = 2.9625$

$t_0 = 0$

$y_1 = 2.9625$ $\quad m = \frac{2.9625 + .2}{1-(2.9625)^2 + .1^2} = -.4072$ $\quad y_2 = -.4072(.1) + 2.9625$

$t_1 = .1$

$= 2.9218$

$y_2 = 2.9218$ $\quad m = \frac{2.9218 + .4}{1 - 2.9218^2 + .2^2} = -.4431$ $\quad y_3 = -.4431(.1) + 2.9218$

$t_2 = .2$

$= 2.8775$

$y_3 = 2.8775$

$t_3 = .3$

$\vdots$

etc.

8. Classify the following differential equations as a) ordinary or partial, b) order, c) linear or nonlinear. (3 points each)

a. $t^3 \frac{dy}{dt} + 2y = \tanh t$

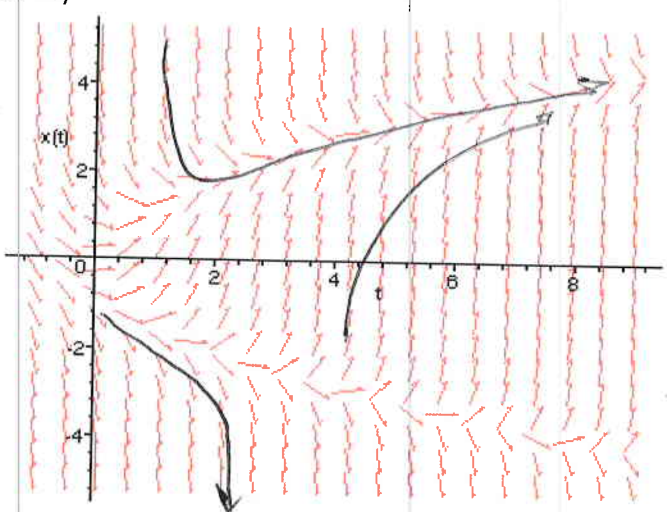
linear, ordinary, 1st order

b. $y' = \frac{3x-y}{y^2-2y}$ nonlinear, first order, ordinary

c. $u_{xxy} + u_y = ye^x$

partial, linear 3rd order

9. Plot three distinct trajectories in the direction field shown below from three distinct initial conditions. (8 points)



10. Solve the second order ordinary differential equations with constant coefficients. (8 points each)

a. $y'' + 2y' - 10y = 0,\ y(1) = 1,\ y'(1) = 5$

$r^2 + 2r - 10 = 0$

$C_1 \approx .138495$
$C_2 \approx -30.31387$

$r = \frac{-2 \pm \sqrt{4+40}}{2} = \frac{-2 \pm \sqrt{44}}{2} = \frac{-2 \pm 2\sqrt{11}}{2} = -1 \pm \sqrt{11}$

$y(t) = c_1 e^{(-1+\sqrt{11})t} + c_2 e^{(-1-\sqrt{11})t}$

$1 = c_1 e^{-1+\sqrt{11}} + c_2 e^{-1-\sqrt{11}}$

$y'(t) = (-1+\sqrt{11}) c_1 e^{(-1+\sqrt{11})t} + c_2(-1-\sqrt{11}) e^{(-1-\sqrt{11})t}$

$y'(1) = 5 = (-1+\sqrt{11}) e^{-1+\sqrt{11}} c_1 + (-1-\sqrt{11}) c_2 e^{-1-\sqrt{11}}$

$\boxed{y(t) = .1385 e^{(-1+\sqrt{11})t} - 30.314^{(-1-\sqrt{11})t}}$

decimals ok for #'s this crazy

b. $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$ $\boxed{r = -2}$

$\boxed{y(t) = c_1 e^{-2t} + c_2 t e^{-2t}}$

c. $y'' + 4y' + 8y = 0$

$r^2 + 4r + 8 = 0$

$r = \frac{-4 \pm \sqrt{16-32}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i$

$\boxed{y(t) = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t}$

11. Find the value of the Wronskian using Abel's Theorem of the differential equation $ty'' - 3(t-1)y' + 6y = 7\sin t$. (6 points)

$$y'' - \frac{3(t-1)}{t} + \frac{6}{t}y = \frac{7}{t}\sin t$$

$$e^{-\int \frac{-3(t-1)}{t}dt} = e^{3\int 1-\frac{1}{t}dt} = e^{3(t-\ln t)}$$

$$= e^{3t}\cdot e^{-3\ln t} = t^{-3}e^{3t} = W = c_1\frac{e^{3t}}{t^3}$$

12. Determine if the solutions $y_1 = t, y_2 = e^{2t}, y_3 = te^{2t}$ form a fundamental set by finding the value of the Wronskian. (7 points)

$$\begin{vmatrix} t & e^{2t} & te^{2t} \\ 1 & 2e^{2t} & e^{2t}+2te^{2t} \\ 0 & 4e^{2t} & \underbrace{2e^{2t}+2e^{2t}+4te^{2t}}_{4e^{2t}+4te^{2t}} \end{vmatrix}$$

it is a fundamental set

$$t\begin{vmatrix} 2e^{2t} & e^{2t}+2te^{2t} \\ 4e^{2t} & 4e^{2t}+4te^{2t} \end{vmatrix} - 1\begin{vmatrix} e^{2t} & te^{2t} \\ 4e^{2t} & 4e^{2t}+4te^{2t} \end{vmatrix} + 0$$

$$t[8e^{4t} + \cancel{8te^{4t}} - 4e^{4t} - \cancel{8te^{4t}}] - [\cancel{4}e^{4t} + \cancel{4te^{4t}} - \cancel{4te^{4t}}] = 4te^{4t} - 4e^{4t} \neq 0$$

13. Find the solution to the Cauchy-Euler equation for the ODE $t^2y'' + ty' + y = 0$. (10 points)

$$t^r = y \quad y' = rt^{r-1} \quad y'' = r(r-1)t^{r-2}$$

$$\cancel{t^2}r(r-1)\cancel{t^{r-2}} + \cancel{t}r\cancel{t^{r-1}} + \cancel{t^r} = 0$$

$$r(r-1) + r + 1 = 0$$

$$r^2 \cancel{-r} \cancel{+r} + 1 = 0$$

$$r^2 + 1 = 0$$

$$r = \pm i$$

$$t^i = e^{i\ln t} = \cos(\ln t) + i\sin(\ln t)$$

$$\boxed{y(t) = c_1\cos(\ln t) + c_2\sin(\ln t)}$$

14. Use the method of reduction of order to find the second solution to the differential equation $xy'' + 2(1+x)y' + 2y = 0$ for the given solution $y_1 = \frac{1}{x}$. (12 points)

$y_2 = \frac{1}{x} v(x)$

$y_2' = -x^{-2} v(x) + \frac{1}{x} v'(x)$

$y_2'' = 2x^{-3} v(x) + -x^{-2} v'(x) - x^{-2} v'(x) + \frac{1}{x} v''(x)$

$2x^{-3} v(x) - 2x^{-2} v'(x) + \frac{1}{x} v''(x)$

$x\left(2x^{-3} v(x) - 2x^{-2} v'(x) + \frac{1}{x} v''(x)\right) +$

$2(1+x)\left(-x^{-2} v(x) + \frac{1}{x} v'(x)\right) + 2\frac{1}{x} v(x) = 0$

$\cancel{2x^{-2} v(x)} - \cancel{\frac{2}{x} v'(x)} + v''(x) + \cancel{-2x^{-2} v(x)} + \cancel{\frac{2}{x} v'(x)} - \cancel{\frac{2}{x} v(x)} + 2v'(x) + \cancel{\frac{2}{x} v(x)} = 0$

$v''(x) + 2v'(x) = 0$ $\quad v'' + 2v' = 0$ $\quad$ let $u = v'$, $u' = v''$

$u' + 2u = 0$

$u' = -2u$

$\int \frac{du}{u} = \int -2\,dx \Rightarrow \ln u = -2x + C$

$u = Ae^{-2x} = v'$

$v = \int Ae^{-2x}\,dx = \frac{A}{-2} e^{-2x} + C$

$y_2 = e^{-2x} \cdot \frac{1}{x}$

15. Find the particular solution to the nonhomogeneous differential equation $y'' + y' + 4y = 2\sin t$ using the method of undetermined coefficients. (12 points)

$r^2 + r + 4 = 0$

$\frac{-1 \pm \sqrt{1 - 16}}{2} = \frac{-1 \pm \sqrt{-15}}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}\, i}{2}$ $\quad y(t) = c_1 e^{-\frac{1}{2}t} \cos \frac{\sqrt{15}}{2} t + c_2 e^{-\frac{1}{2}t} \sin \frac{\sqrt{15}}{2} t$

$Y(t) = A\sin t + B\cos t$

$Y'(t) = A\cos t - B\sin t$

$Y''(t) = -A\sin t - B\cos t$

$-A\sin t - B\cos t + A\cos t - B\sin t + 4A\sin t + 4B\cos t = 2\sin t$

Sines

$-A + 4A - B = 2 \Rightarrow 3A - B = 2$

Cosines

$-B + A + 4B = 0 \Rightarrow 3B + A = 0$

$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ 1 & 3 & 0 \end{array}\right]$

$A = 3/5 \quad B = -1/5$

$\boxed{y(t) = c_1 e^{-\frac{1}{2}t} \cos \frac{\sqrt{15}}{2} t + c_2 e^{-\frac{1}{2}t} \sin \frac{\sqrt{15}}{2} t + \frac{3}{5}\sin t - \frac{1}{5}\cos t}$

16. Suppose that the solutions to a second order differential equation are $y_1(t) = e^{-t}, y_2(t) = e^{3t}$. If the forcing term on the nonhomogeneous ODE is $F(t) = t^2e^{-t} + e^{2t} + \frac{1}{2}\cos t$, state your initial Ansatz for the method of undetermined coefficients (you do not need to solve for any of the coefficients, just state where you would start). (8 points)

$(At^2+Bt+C)te^{-t} = (At^3+Bt^2+Ct)e^{-t} \rightarrow$ for t^2e^{-t}

$De^{2t} \rightarrow$ for e^{2t}

$E\sin t + F\cos t \rightarrow$ for $\frac{1}{2}\cos t$

$Y(t) = (At^3+Bt^2+Ct)e^{-t} + De^{2t} + E\sin t + F\cos t$

17. Find the solution to the boundary value problem $y'' + y = 0, y'(0) = 1, y(L) = 0$. (8 points)

$r^2 + 1 = 0$

$r = \pm i$

$y = c_1\cos t + c_2\sin t$

$y' = c_1\sin t + c_2\cos t$

$1 = c_1(0) + (+c_2)(1)$

$\Rightarrow 1 = +c_2$

$c_2 = +1$

$0 = c_1\cos L + \sin L$

$-\sin L = c_1\cos L$

$-\frac{\sin L}{\cos L} = c_1$

$-\tan L = c_1$

$\boxed{y(t) = (-\tan L)\cos t + \sin t}$