Math 2153, Final Exam, Summer 2015 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Find the mass of the wire described by the path and given density function $\vec{r}(t) = 3\cos(t)\,\hat{\imath} + 2t\hat{\jmath} + \sin(t)\,\hat{k}, \rho(x, y, z) = kz, 0 \le t \le \pi$ by calculating $\int_c \rho(x, y, z)ds$. (12 points)

$z = \sin t$

$r'(t) = 3\sin t\,\hat{\imath} + 2\hat{\jmath} + \cos t\,\hat{k}$

$\|r'(t)\| = \sqrt{9\sin^2 t + 4 + \cos^2 t}$

$= \sqrt{8\sin^2 t + \sin^2 t + 4\cos^2 t} = \sqrt{8\sin^2 t + 5}$

$k\int_0^{\pi} \sin t\sqrt{8\sin^2 t + 5}\, dt$

$\approx 6.38k$

$ds = \|r'(t)\|dt$

2. Find the velocity, speed, acceleration and jerk of a particle traveling along the path $\vec{r}(t) = (t - \sin t)\vec{i} + (1 - \cos t)\vec{j}, (\pi, 2)$. If a specific point is given, evaluate each derivative at the specified point. (12 points)

$t = \pi$

$r'(t) = v(t) = (1 - \cos t)\hat{\imath} + (\sin t)\hat{\jmath}$ $\quad v(\pi) = 2\hat{\imath} + 0\hat{\jmath}$

$r''(t) = a(t) = (\sin t)\hat{\imath} + \cos t\,\hat{\jmath}$ $\quad a(\pi) = 0\hat{\imath} - \hat{\jmath}$

$r'''(t) = j(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$ $\quad j(\pi) = -\hat{\imath} + 0\hat{\jmath}$

$\|r'(t)\| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t}$

$\|r'(\pi)\| = \sqrt{2^2} = 2$

3. Find the acceleration, velocity and position function for a particle with $\vec{a}(t) = e^t\hat{\imath} - 8\hat{k}, \vec{v}(0) = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}, \vec{r}(0) = \vec{0}$. (12 points)

$v(t) = \int a(t)\,dt = (e^t + C_1)\hat{\imath} + C_2\hat{\jmath} + (-8t + C_3)\hat{k}$

$1 + C_1 = 2 \Rightarrow C_1 = 1 \quad C_2 = 3 \quad -8t + C_3 = 1 \Rightarrow C_3 = 1$

$v(t) = (e^t + 1)\hat{\imath} + 3\hat{\jmath} + (-8t + 1)\hat{k}$

$r(t) = \int v(t)\,dt = (e^t + t + C_1)\hat{\imath} + (3t + C_2)\hat{\jmath} + (-\frac{8}{2}t^2 + t + C_3)\hat{k}$

$e^0 + 0 + C_1 = 0 \Rightarrow C_1 = -1 \quad C_2 + 3(0) = 0 \Rightarrow C_2 = 0$

$-4(0)^2 + 9(0) + C_3 = 0 \Rightarrow C_3 = 0$

$r(t) = (e^t + t - 1)\hat{\imath} + 3t\hat{\jmath} + (-4t^2 + t)\hat{k}$

4. Use the method of Lagrange multipliers to maximize the function $f(x, y, z, w) = 3x^2 + y^2 + 2z^2 - 5w^2$ subject to the constraint to $x + 6y + 3z + 2w = 4$. (12 points)

$g = x + 6y + 3z + 2w - 4$

$\nabla f = \langle 6x, 2y, 4z, -10w \rangle \quad \lambda \nabla g = \langle \lambda, 6\lambda, 3\lambda, 2\lambda \rangle$

$6x = \lambda \quad 2y = 6\lambda \quad 4z = 3\lambda \quad -10w = 2\lambda$

$x = \lambda/6 \quad y = 3\lambda \quad z = 3/4\,\lambda \quad w = -1/5\,\lambda$

$\frac{1}{6}\lambda + 6(3\lambda) + 3(3/4\,\lambda) + 2(-1/5\,\lambda) = 4$

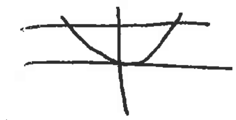
$\frac{1}{6}\lambda + 18\lambda + 9/4\,\lambda + -2/5\,\lambda = 4$

$\frac{1201}{60}\lambda = 4 \quad \Rightarrow \lambda = \frac{240}{1201}$

$x = \frac{240}{1201} \cdot \frac{1}{6} = \frac{40}{1201} \quad y = 3\left(\frac{240}{1201}\right) = \frac{720}{1201}$

$z = \frac{240}{1201} \cdot \frac{3}{4} = \frac{180}{1201} \quad w = -\frac{1}{5}\left(\frac{240}{1201}\right) = \frac{-48}{1201}$

5. Find the average value of the function $\bar{f} = \frac{1}{A}\int\int_R f(x,y)dA$ for $f(x,y) = sin^2(x)$ over the region bounded by $y = x^2, y = 4$. (15 points)



$$A = \int_{-2}^{2}\int_{x^2}^{4} dy\,dx = \int_{-2}^{2} 4 - x^2\,dx$$

$$= 2\int_0^2 4 - x^2 dx = 2\left[4x - \tfrac{1}{3}x^3\Big|_0^2\right] = 2\left[8 - \tfrac{8}{3}\right] = (16/3)2 = \frac{32}{3}$$

$$\frac{1}{A}\int_{-2}^{2}\int_{x^2}^{4} \sin^2 x\,dy\,dx = \frac{3}{32}\int_{-2}^{2}\int_{x^2}^{4} \tfrac{1}{2}(1 - \cos 2x)\,dy\,dx =$$

$$\frac{3}{64}\int_{-2}^{2} y(1 - \cos 2x)\Big|_{x^2}^{4} dx = \frac{3}{64}\int_{-2}^{2} 4(1 - \cos 2x) - x^2(1 - \cos 2x)\,dx =$$

$$\frac{3}{64}\int_{-2}^{2} 4 - 4\cos 2x - x^2 + x^2\cos 2x\,dx = \frac{3}{32}\int_0^2 4 - 4\cos 2x - x^2 + x^2\cos 2x\,dx$$

$$= \frac{3}{32}\left[4x - 2\sin 2x - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\sin 2x + \tfrac{1}{2}x\cos 2x - \tfrac{1}{4}\sin 2x\Big|_0^2\right.$$

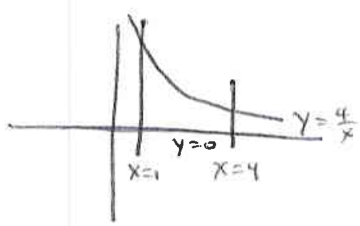
$$= \frac{3}{32}\left[8 - \cancel{2\sin 4} - \tfrac{8}{3} + \cancel{2\sin 4} + \cos 4 - \tfrac{1}{4}\sin 4\right] = \frac{3}{32}\left[\tfrac{16}{3} + \cos 4 - \tfrac{1}{4}\sin 4\right]$$

$\approx .4697$

u	dv
$+x^2$	$\cos 2x$
$-2x$	$\frac{1}{2}\sin 2x$
$+2$	$-\frac{1}{4}\cos 2x$
-0	$-\frac{1}{8}\sin 2x$

6. Find the center of mass of the lamina defined by $xy = 4, x = 1, x = 4, y = 0, \rho = kx^2$. (16 points)

$y = \frac{4}{x}$



$$M = \int_1^4\int_0^{4/x} kx^2\,dy\,dx = k\int_1^4 x^2 y\Big|_0^{4/x} dx =$$

$$\int_1^4 k\,x^2\cdot\frac{4}{x}\,dx = 4k\int_1^4 x\,dx = 4k\cdot\tfrac{1}{2}x^2\Big|_1^4$$

$$\tfrac{4}{2}k[16 - 1] = 30k$$

$$M_x = \int_1^4\int_0^{4/x} ky\,x^2\,dy\,dx = k\int_1^4 \tfrac{1}{2}y^2x^2\Big|_0^{4/x} dx = k\int_1^4 \tfrac{1}{2}\left(\tfrac{4}{x}\right)^2 x^2\,dx =$$

$$8k\int_1^4 dx = 8k(x)\Big|_1^4 = 8k(4-1) = 24k$$

$$M_y = \int_1^4\int_0^{4/x} x\cdot kx^2\,dy\,dx = k\int_1^4 y\,x^3\Big|_0^{4/x} dx = k\int_1^4 \frac{4}{x}\cdot x^3\,dx = 4k\int_1^4 x^2\,dx =$$

$$\tfrac{4}{3}k\,x^3\Big|_1^4 = \tfrac{4}{3}k[64 - 1] = 84k$$

Center of mass

$$\bar{x} = \frac{84k}{30k} = \frac{M_y}{M} = \frac{14}{5} \qquad \bar{y} = \frac{24k}{30k} = \frac{M_x}{M} = \frac{4}{5}$$

$\left(\frac{14}{5}, \frac{4}{5}\right)$

7. Find a parametric equation for the curve created from the intersection of surfaces $z = x^2 + y^2, x + y = 0 \quad x = t$. (8 points)

$t + y = 0 \Rightarrow y = -t$

$z = t^2 + (-t)^2 = 2t^2$

$r(t) = t\hat{\imath} - t\hat{\jmath} + 2t^2\hat{k}$

8. Find a parametric (vector-valued) function for the surface given by $\frac{4x^2 + y^2 = 16}{16}$. (8 points)

$\vec{r}(u,v) = 2\cos u\hat{\imath} + 4\sin u\hat{\jmath} + v\hat{k}$

$\frac{x^2}{4} + \frac{y^2}{16} = 1$

$b=2 \quad a=4$

9. Find the limits, if they exist, or prove that they do not. You will need to check multiple paths. You may need to use polar or spherical coordinates. (10 points each)

a. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^3 + y^2}$

path $x^3 = y^2$

$y = x^{3/2}$

$\lim\limits_{kx\to 0} \frac{x^2 \cdot kx^{3/2}}{x^3 + k^2x^3} = \lim\limits_{x\to 0} \frac{kx^{7/2}}{x^3(1+k^2)} = \lim\limits_{x\to 0} \frac{kx^{1/2}}{1+k^2} \to 0$

path $y = kx$

$\lim\limits_{x\to 0} \frac{x^2 \cdot kx}{x^3 + k^2x^2} = \lim\limits_{x\to 0} \frac{kx^3}{x^2(x+k^2)} = \lim\limits_{x\to 0} \frac{kx}{x+k^2} = 0$

limit appears to be 0.

b. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{3x^2 + 2y^2}$

path $y = kx$

$\lim\limits_{x\to 0} \frac{xk^2x^2}{3x^2 + 2k^2x^2} = \lim\limits_{x\to 0} \frac{kx^3}{x^2(3+2k^2)} = \lim\limits_{x\to 0} \frac{kx}{3+2k^2} = 0$

limit appears to be zero (0).

10. For the function $s(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$, draw the gradient field below by indicating any curves where the partial derivatives are zero, and at least one vector of the field in each region of the graph. Use that to determine if any critical points are maxima, minima or saddle points. (14 points)

$y = -\frac{1}{2}x + 1$

$\frac{\partial s}{\partial x} = 2x + 4y - 4 = 0 \Rightarrow x + 2y = 2$

$\frac{\partial s}{\partial y} = 4x + 2y + 16 = 0 \Rightarrow 2x + y = -8 \Rightarrow y = -2x - 8$

$x + 2(-2x - 8) = 2 \Rightarrow$

$x - 4x - 16 = 2$

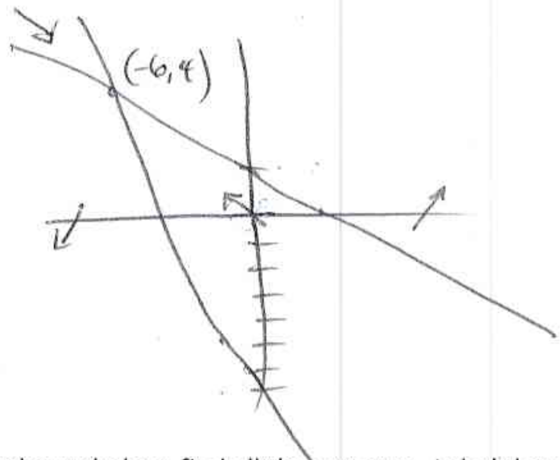
$-3x = 18$

$x = -6$

$y = -2(-6) - 8 = 4$

$(-6, 4)$

Saddle point at $(-6, 4)$



$\nabla f = \langle 2x + 4y - 4, 4x + 2y + 16 \rangle$

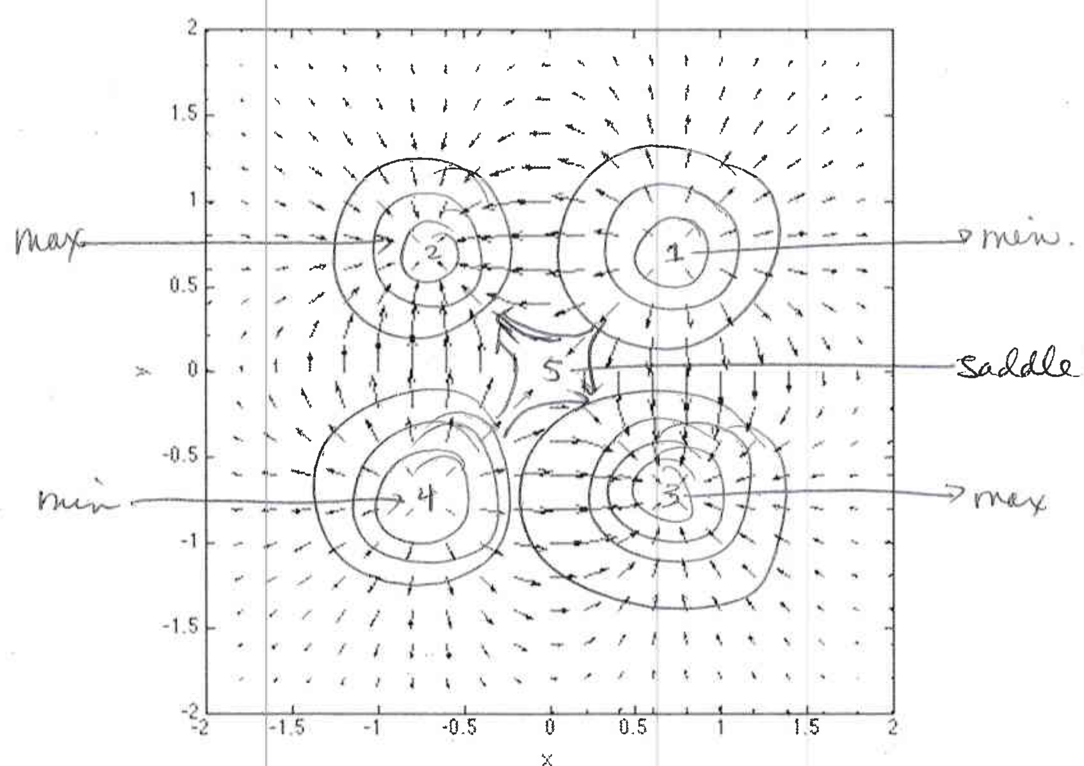
$\nabla f(0,0) = \langle -4, 16 \rangle$

$\nabla f(4,0) = \langle 4, 24 \rangle$

$\nabla f(-6,0) = \langle -16, -8 \rangle$

$\nabla f(-7,5) = \langle 2, -2 \rangle$

11. For the gradient field shown below, find all the extrema. Label them as maxima, minima, or saddle points. Draw 10 level curves on the graph. (10 points)



12. Use the second partials test to find the extrema of the function $f(x,y) = 2x^4 - 6x^2y + y^2$. Classify all the critical points as a minimum, maximum or a saddle point. If this is not possible with the second partials test, explain why not. (18 points)

$f_x = \frac{\partial f}{\partial x} = 8x^3 - 12xy = 0 \Rightarrow 2x^3 - 3xy = 0 \quad x(2x^2 - 3y) = 0 \quad x = 0$

$2x^2 = 3y \Rightarrow -6x^2 = -9y$

$f_y = \frac{\partial f}{\partial y} = -6x^2 + 2y = 0 \Rightarrow -9y + 2y = 0 \Rightarrow -7y = 0 \quad y = 0$

$(0,0)$

$f_{xx} = 24x^2 - 12y = 0$

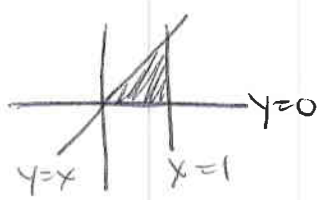
$f_{yy} = 2$

$f_{xy} = -12x = 0$

$f_{xx} f_{yy} - (f_{xy})^2 = 0 \cdot 2 - 0 = 0$

Cannot be determined / 2nd partials test in conclusive.

13. Set up a double integral to find the volume of the solid bounded by the graphs: $z = xy, z = 0, y = x, x = 1$. Find the value of the integral. [Hint: a sketch of the region may help you.] (15 points)



$\int_0^1 \int_0^x xy \, dy \, dx$

$\int_0^1 \frac{1}{2} y^2 \cdot x \Big|_0^x = \frac{1}{2} \int_0^1 x^3 \, dx = \frac{1}{8} \int_0 x^4 \Big|_0^1 = \frac{1}{8}$

14. Set up a double integral to find the volume of the solid bounded by the graphs, in polar coordinates: $z=\ln(x^2+y^2), z=0, x^2+y^2\geq 1, x^2+y^2\leq 4$. Find the value of the integral. (15 points)

$z = \ln r^2 = 2\ln r$

$$\int_0^{2\pi}\int_1^4 2\ln r \cdot r\,dr\,d\theta$$

$u=\ln r \quad dv = r\,dr$

$du = \frac{1}{r}dr \quad v=\frac{1}{2}r^2.$

$\int -\frac{1}{2}r$

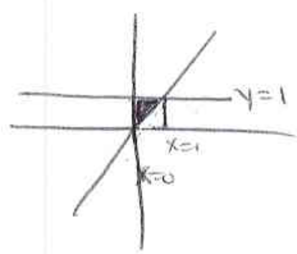
$$2\int_0^{2\pi}\left[\frac{1}{2}r^2\ln r - \frac{1}{4}r^2\Big|_1^4 d\theta = 2\int_0^{2\pi}\frac{1}{2}\cdot 16\ln 4 - \frac{1}{4}16 - \left(\cancel{\frac{1}{2}\ln 1} + \frac{1}{4}(1)\right)d\theta = 2\int_0^{2\pi} 8\ln 4 - \frac{15}{4}\,d\theta$$

$$4\pi\left[8\ln 4 - \frac{15}{4}\right] \approx 92.24$$

15. Change the order of integration and find the value of the integral $\int_0^1\int_x^1 x\sqrt{1+2y^3}\,dy\,dx$. (12 points)

$y=1$

$y=x$



$$\int_0^1\int_0^y x\sqrt{1+2y^3}\,dx\,d\ =$$

$$\int_0^1 \frac{1}{2}x^2\sqrt{1+2y^3}\Big|_0^y dy = \int_0^1 \frac{1}{2}y^2\sqrt{1+2y^3}\,dy$$

$u = 1+2y^3$

$du = 6y^2\,dy$

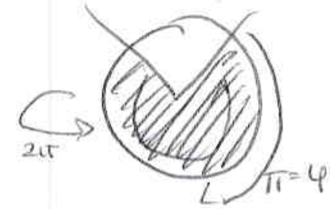
$\frac{1}{6}du = y^2\,dy$

$$\frac{1}{2}\int \cdot\frac{1}{6}u^{1/2}\,du = \frac{2}{3}\cdot\frac{1}{12}(1+2y^3)^{3/2}\Big|_0^1 =$$

$$\frac{1}{18}(3^{3/2} - 1^{3/2}) = \frac{3^{3/2}}{18} = \frac{3\sqrt{3}}{18} = \boxed{\frac{\sqrt{3}}{6} - \frac{1}{18}}$$

16. Find a triple integral to calculate the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cone $z=\sqrt{x^2+y^2}$. Find the value of the integral. [Hint: This will be easiest to evaluate in spherical coordinates.] (15 points)

$$\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^4 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$



$$\frac{1}{3}\rho^3\Big|_0^4 = \frac{64}{3}$$

$$\int_0^{2\pi}\int_{\pi/4}^{\pi} \sin\varphi \cdot \frac{64}{3}\, d\varphi \, d\theta = \frac{64}{3}\int_0^{2\pi} -\cos\varphi\Big|_{\pi/4}^{\pi} d\theta = \frac{64}{3}\int_0^{2\pi} -(-1)-(-1/\sqrt{2})\, d\theta$$

$$= \frac{64}{3}\,(1+1/\sqrt{2})\cdot 2\pi = \frac{128\pi}{3}\left(1+\frac{\sqrt{2}}{2}\right) = \frac{64\pi(2+\sqrt{2})}{3}$$

17. For the function $\vec{r}(t) = 3\sin 2t\,\hat{\imath} + 3\cos 2t\,\hat{\jmath} + 4t\hat{k}$, find the equation of the tangent line at $t=0$. Write your answer in parametric form. (12 points)

$$r'(t) = 6\cos 2t\,\hat{\imath} + -6\sin 2t\,\hat{\jmath} + 4\hat{k}$$

$$r'(0) = 6\hat{\imath} - 0\hat{\jmath} + 4\hat{k} \qquad \langle a, b, c\rangle = \langle 6, 0, 4\rangle$$

$$r(0) = 0\hat{\imath} + 3\hat{\jmath} + 0\hat{k} \qquad 0 = x_0,\ 3 = y_0,\ 0 = z_0$$

$$l(t) = (0+6t)\hat{\imath} + (3+0t)\hat{\jmath} + (0+4t)\hat{k} = 6t\hat{\imath} + 3\hat{\jmath} + 4t\hat{k}$$

18. Find the equation of the tangent plane to the equation $y\ln xz^2 = 2, P(e,2,1)$ at the specified point. (10 points)

$G = y\ln xz^2 - 2 \quad = y\ln x + 2y\ln z - 2$

$\nabla G = \langle y/x,\ \ln xz^2,\ \frac{2y}{z} \rangle$

$\nabla G(e,2,1) = \quad \langle \frac{2}{e}, 1 \quad , 4 \rangle$

$\frac{2}{e}(x-e) + 1(y-2) + 4(z-1) = 0$

19. Find the value of the line integral in the specified field over the specified curve $\int_C \vec{F}\cdot d\vec{r}, F(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{k}, C: line\ from\ (0,0,0)\ to\ (5,3,2)$. Check that the field is conservative, and if it is, use the fundamental theorem of line integrals to find the value of the integral. (12 points)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x-x)\hat{i} + (y-y)\hat{j} + (z-z)\hat{k} = \vec{0}$$

yes, it's conservative

$\int yz\,dx = xyz + \text{stuff}$

$\int xz\,dy = xyz + \text{stuff}$

$\int xy\,dz = xyz + \text{stuff}$

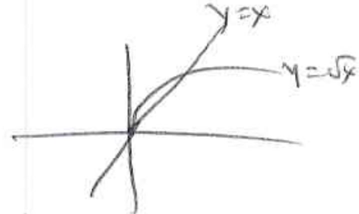
$f(x,y,z) = xyz$

$\int_C \vec{F}\cdot d\vec{r} =$

$5\cdot 3\cdot 2 - 0\cdot 0\cdot 0 = \boxed{30}$

20. Evaluate the line integral $\oint_C \overset{M}{2xy}dx + \overset{N}{(x+y)}dy$ around the region bounded by the curves $y = x, y = \sqrt{x}$ using Green's Theorem. (12 points)

$$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial M}{\partial y} = 2x$$

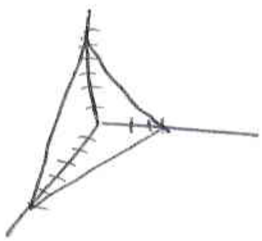


$$\int_0^1 \int_x^{\sqrt{x}} (1-2x)\,dy\,dx = \int_0^1 y(1-2x)\Big|_x^{\sqrt{x}} dx = \int_0^1 \sqrt{x} - 2x^{3/2} - x + 2x^2\,dx =$$

$$\int_0^1 x^{1/2} - 2x^{3/2} - x + 2x^2\,dx = \frac{2}{3}x^{3/2} - \frac{4}{5}x^{5/2} - \frac{1}{2}x^2 + \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3} - \frac{4}{5} - \frac{1}{2} + \frac{2}{3} = \boxed{\frac{1}{30}}$$

21. Use the Divergence Theorem to find the flux through the closed surface $\vec{F}(x,y,z) = (2x - y)\hat{\imath} + (z - 2y)\hat{\jmath} + z\hat{k}, S: plane\ 2x + 4y + 2z = 12, z = 0, x = 0, y = 0.$ (15 points)

$$\iiint_R \vec{\nabla}\cdot\vec{F}\,dV \qquad \vec{\nabla}\cdot\vec{F} = 2 - 2 + 1 = 1$$



$2z = 12 - 2x - 4y$

$z = 6 - x - 2y$

$2x + 4y = 12$

$4y = 12 - 2x$

$y = 3 - \frac{1}{2}x$

$$\int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{6-x-2y} 1\,dz\,dy\,dx$$

$$\int_0^6 \int_0^{3-\frac{1}{2}x} 6 - x - 2y\,dy\,dx = \int_0^6 6y - xy - y^2\Big|_0^{3-\frac{1}{2}x} dx$$

$$= \int_0^6 6(3 - \tfrac{1}{2}x) - x(3 - \tfrac{1}{2}x) - (3 - \tfrac{1}{2}x)^2\,dx = \int_0^6 18 - 3x - \cancel{3x} + \tfrac{1}{2}x^2 - 9 + \cancel{3x} - \tfrac{1}{4}x^2\,dx$$

$$\int_0^6 9 - 3x + \frac{1}{4}x^2\,dx = 9x - \frac{3}{2}x^2 + \frac{1}{12}x^3\Big|_0^6 = 54 - 54 + 18 = \boxed{18}$$