Math 2153, Exam #3, Summer 2015 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Write an expression (an integral) for the length of the curve $\vec{r}(t)=\vec{i}+t^2\vec{j}+t^3\vec{k};[0,2]$ on the given interval. Evaluate the integral numerically and round your answer to 4 decimal places. (7 points)

$r'(t) = 2t\hat{j} + 3t^2\hat{k}$ $\quad \|r'(t)\| = \sqrt{4t^2+9t^4} = t\sqrt{4+9t^2}$

$\int_0^2 t\sqrt{4+9t^2}\,dt$ $\quad u = 4+9t^2$, $du = 18t\,dt$, $\frac{1}{18}du = t\,dt$

$\int \frac{1}{18} u^{1/2}\,du =$

$\frac{1}{18}\cdot\frac{2}{3}u^{3/2} = \frac{1}{\cancel{18}_{9}}\cdot\frac{\cancel{2}}{3}(4+9t^2)^{3/2}\Big|_0^2$

$\frac{1}{27}\left[40^{3/2} - 4^{3/2}\right] \approx 9.0734$

2. For the function $\vec{r}(t) = 3\sin 2t\,\hat{\imath} + 3\cos 2t\,\hat{\jmath} + 4t\hat{k}$, find the following: (8 points each)
   a. The unit tangent vector

$r'(t) = 6\cos 2t\,\hat{\imath} + {}^{-}6\sin 2t\,\hat{\jmath} + 4\hat{k}$ $\quad \|r'(t)\| = \sqrt{36\cos^2 2t + 36\sin^2 2t + 16} = \sqrt{52} = 2\sqrt{13}$

$\frac{r'(t)}{\|r'(t)\|} = \vec{T}(t) = \left(\frac{3\cos 2t}{\sqrt{13}}\right)\hat{\imath} - \left(\frac{3\sin 2t}{\sqrt{13}}\right)\hat{\jmath} + \frac{2}{\sqrt{13}}\hat{k}$

   b. The unit normal vector

$T'(t) = \frac{-6}{\sqrt{13}}\sin 2t\,\hat{\imath} - \frac{6}{\sqrt{13}}\cos 2t\,\hat{\jmath}$ $\quad \|T'(t)\| = \sqrt{\frac{36}{13}\sin^2 2t + \frac{36}{13}\cos^2 2t} = \sqrt{\frac{36}{13}} = \frac{6}{\sqrt{13}}$

$N(t) = \frac{T'(t)}{\|T'(t)\|} = -\sin 2t\,\hat{\imath} - \cos 2t\,\hat{\jmath}$

   c. Find the equation of the tangent line at $t = 0$. Write your answer in parametric form.

$\vec{r}(0) = 0\hat{\imath} + 3\hat{\jmath} + 0\hat{k}$ $\quad r'(0) = 6\hat{\imath} - 0\hat{\jmath} + 4\hat{k}$

tang. line $\vec{r}_2(t) = 6t\hat{\imath} - 3\hat{\jmath} + 4t\hat{k}$

3. Find the directional derivative of the function $f(x,y,z) = xy + yz + xz; P(1,1,1); \vec{v} = 2\vec{i} + \vec{j} - \vec{k}$ in the specified direction, at the specified point. (8 points)

$\|\vec{v}\| = \sqrt{2^2+1^2+1^2} = \sqrt{4+1+1} = \sqrt{6}$ $\quad \vec{u} = \frac{2}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$

$\nabla f = \langle y+z, x+z, x+y \rangle$ at $P(1,1,1)$ $\nabla f = \langle 2,2,2 \rangle$

$\nabla f \cdot \vec{u} = 2 \cdot \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} = \boxed{\frac{4}{\sqrt{6}}}$

4. Use the function above to find the direction in which the directional derivative is a maximum. (5 points)

in direction of the gradient

$\langle 2,2,2 \rangle$ (at this point)

or ∇f at any point

5. Find the equation of the tangent plane to the equation $y \ln xz^2 = 2, P(e,2,1)$ at the specified point. (8 points)

$F(x,y,z) = y \ln xz^2 - 2 \Rightarrow$ $\quad \nabla F = \langle \frac{y}{xz^2} \cdot z^2, \ln xz^2, \frac{y}{xz^2} \cdot x \cdot 2z \rangle$

$= \langle \frac{y}{x}, \ln xz^2, \frac{2y}{z} \rangle$

$\nabla F(e,2,1) = \langle \frac{2}{e}, 1, \frac{4}{1} \rangle$

tangent plane:

$\frac{2}{e}(x-e) + 1(y-2) + 4(z-1) = 0$

6. Find the equation of the tangent plane to the surface at the specified point.
$\vec{r}(u,v) = 2\cos u\vec{i} + v\vec{j} + 2\sin u\vec{k}, P(2,4,0)$ (9 points)

$2\cos u = 2 \quad u = 0$
$v = 4$

$\vec{r}_u = -2\sin u\,\hat{i} + 0\hat{j} + 2\cos u\,\hat{k} \qquad \vec{r}_u(0,4) = 0\hat{i} + 0\hat{j} + 2\hat{k}$

$\vec{r}_v = 0\hat{i} + 1\hat{j} + 0\hat{k} \qquad \vec{r}_v(0,4) = 0\hat{i} + \hat{j} + 0\hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\hat{i} - 0\hat{j} + 0\hat{k}$$

$-2(x-2) + 0(y-4) + 0(z-0) = 0$

or $x - 2 = 0$

7. Find the total differential of the function $g(x,y,z) = x^2yz^2 + \sin yz$ at the point $g(1,2,0)$ and use it to approximate the value of the function at $g(1.1, 1.8, 0.3)$. (8 points)

$dg = f_x\,dx + f_y\,dy + f_z\,dz \qquad f(1,2,0) = 0$

$f_x = 2xyz^2 \qquad f_x(1,2,0) = 0$

$f_y = x^2z^2 + z\cos yz \qquad f_y(1,2,0) = 0 + 0 = 0$

$f_z = 2x^2yz + y\cos yz \qquad f_z(1,2,0) = 0 + 2\cos 0 = 2$

$dx = 0.1,\ dy = -.2,\ dz = .3$

$dg = 0(.1) + 0(-.2) + 2(.3) = \boxed{.6} \qquad \boxed{f(1.1, 1.8, .3) \approx 0 + .6 = .6}$

8. Find the curvature of the function $\vec{r}(t) = \sin t\,\hat{\imath} + \sin 2t\,\hat{\jmath}$. Evaluate it at $t = \frac{\pi}{4}$. (10 points)

$$K = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$r'(t) = \cos t\,\hat{i} + 2\cos 2t\,\hat{j}$
$r''(t) = -\sin t\,\hat{i} - 4\sin 2t\,\hat{j}$

$$r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & 2\cos 2t & 0 \\ -\sin t & -4\sin 2t & 0 \end{vmatrix} = (0)\hat{i} - 0\hat{j} + (-4\cos t\sin 2t + 2\cos 2t\sin t)\hat{k}$$

$\sin 2t = 2\sin t\cos t$
$\cos 2t = 1 - 2\cos^2 t$

$\|r'(t) \times r''(t)\| = (-4\cos t \cdot 2\sin t\cos t + 2\sin t(1 - 2\cos^2 t)) =$
$-8\cos^2 t\sin t + 2\sin t - 4\sin t\cos^2 t =$
$|2\sin t - 12\cos^2 t\sin t| =$

$\|r'(t)\| = \sqrt{\cos^2 t + 4\cos^2 2t}$

at $t = \pi/4$

$$K = \frac{|2\sin t - 12\cos^2 t\sin t|}{(\cos^2 t + 4\cos^2 2t)^{3/2}} = \frac{\left|\frac{2}{\sqrt{2}} - \cancel{12}^{6} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\right|}{(\frac{1}{2} + 4 \cdot 0)^{3/2}} = \frac{\left|\frac{-4}{\sqrt{2}}\right|}{(1/2)^{3/2}} = \frac{\frac{4}{\sqrt{2}}}{\frac{1}{2\sqrt{2}}} = \frac{4}{\cancel{\sqrt{2}}} \cdot 2\cancel{\sqrt{2}} = \boxed{8}$$

9. Set up a double integral to find the surface area of the surface $\vec{r}(u,v) = 2u\cos v\,\hat{\imath} + 2u\sin v\,\hat{\jmath} + u^2\hat{k}, 0 \le u \le 2, 0 \le v \le 2\pi$ over the indicated region. You do not need to integrate it. (8 points)

$$\vec{r}_u = 2\cos v\,\hat{\imath} + 2\sin v\,\hat{\jmath} + 2u\hat{k} \qquad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\cos v & 2\sin v & 2u \\ -2u\sin v & 2u\cos v & 0 \end{vmatrix} =$$

$$\vec{r}_v = -2u\sin v\,\hat{\imath} + 2u\cos v\,\hat{\jmath} + 0k$$

$$(0 - 4u^2\cos v)\hat{\imath} - (0 + 4u^2\sin v)\hat{\jmath} + (4u\cos^2 v + 4u\sin^2 v)\hat{k} = -4u^2\cos v\,\hat{\imath} - 4u^2\sin v\,\hat{\jmath} + 4u\hat{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{16u^4\cos^2 v + 16u^4\sin^2 v + 16u^2} = \sqrt{16u^4 + 16u^2} = 4u\sqrt{u^2+1}$$

$$\boxed{S = \int_0^2 \int_0^{2\pi} 4u\sqrt{u^2+1}\,dv\,du}$$

10. Find the value of the line integral in the specified field over the specified curve $\int_C \vec{F} \cdot d\vec{r}, F(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{k}, C: line\ from\ (0,0,0)\ to\ (5,3,2)$. Check that the field is conservative, and if it is, use the fundamental theorem of line integrals to find the value of the integral. (12 points)

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x - x)\hat{\imath} - (y - y)\hat{\jmath} + (z - z)\hat{k} = \vec{0}$$

it is conservative

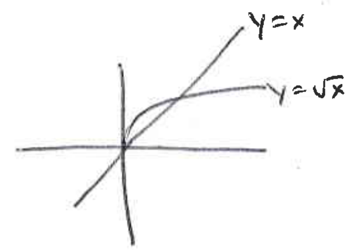
$$\int yz\,dx = xyz + g(y,z)$$
$$\int xz\,dy = xyz + h(x,z)$$
$$\int xy\,dz = xyz + i(x,y)$$

$f(x,y,z) = xyz$ (+ K optional here)

$$\int_C \vec{F} \cdot d\vec{r} = 5 \cdot 3 \cdot 2 - 0 \cdot 0 \cdot 0 = \boxed{30}$$

11. Evaluate the line integral $\int_C \overset{M}{2xy}\,dx + \overset{N}{(x+y)}\,dy$ around the region bounded by the curves $y = x, y = \sqrt{x}$ using Green's Theorem. (8 points)

$$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1-2x$$



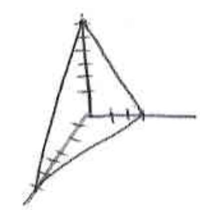
$$\int_0^1 \int_x^{\sqrt{x}} 1-2x \, dy\, dx = \int_0^1 y(1-2x)\Big|_x^{\sqrt{x}} dx =$$

$$\int_0^1 \sqrt{x} - 2x\sqrt{x} - x + 2x^2 \, dx = \int_0^1 x^{1/2} - 2x^{3/2} - x + 2x^2 \, dx =$$

$$\frac{2}{3}x^{3/2} - \frac{2}{5}\cdot 2x^{5/2} - \frac{1}{2}x^2 + \frac{2}{3}x^3 \Big|_0^1 \quad \frac{2}{3} - \frac{4}{5} - \frac{1}{2} + \frac{2}{3} = \boxed{1/30}$$

12. Use the Divergence Theorem to find the flux through the closed surface $\vec{F}(x,y,z) = (2x-y)\hat{\imath} + (z-2y)\hat{\jmath} + z\hat{k}, S: plane\ 2x + 4y + 2z = 12, z = 0, x = 0, y = 0.$ (8 points)

$$\iiint \vec{\nabla}\cdot\vec{F}\, dV \qquad \vec{\nabla}\cdot\vec{F} = 2 - 2 + 1 = 1$$



$$\int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{6-x-2y} dz\, dy\, dx =$$

$$\int_0^6 \int_0^{3-\frac{1}{2}x} z \Big|_0^{6-x-2y} dy\, dx =$$

$$\int_0^6 \int_0^{3-\frac{1}{2}x} 6-2y-x \, dy\, dx =$$

$$\int_0^6 6y - y^2 - xy \Big|_0^{3-\frac{1}{2}x} dx =$$

$$\int_0^6 6(3-\tfrac{1}{2}x) - (3-\tfrac{1}{2}x)^2 - x(3-\tfrac{1}{2}x)\, dy =$$

$$\int_0^6 18 - 3x - (9 - 3x + \tfrac{1}{4}x^2) - 3x + \tfrac{1}{2}x^2 \, dx =$$

$$\int_0^6 9 - 3x + \tfrac{1}{4}x^2 \, dx = 9x - \tfrac{3}{2}x^2 + \tfrac{1}{12}x^3 \Big|_0^6 =$$

$$54 - \tfrac{3}{2}\cdot 36 + \tfrac{1}{12}\cdot 216 = \quad 18$$

$2z = 12 - 2x - 4y$

$z = 6 - x - 2y$

$2x + 4y + \cancel{2(0)} = 12$

$2x + 4y = 12$

$4y = 12 - 2x$

$y = 3 - \frac{1}{2}x$

$2x + 4(0) = 12$

$2x = 12$

$x = 6$

13. Use Stokes' Theorem to find the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$ through the specified field and over the given path $\vec{F}(x,y,z) = x^2\hat{\imath} + z^2\hat{\jmath} - xyz\hat{k}, S: z = \sqrt{4 - x^2 - y^2}, z = 0$. (8 points)

$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$

∇G

top half of ~~s~~phere
radius 2

$z=0$ plane $\vec{n} = \hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz - 2z)\hat{\imath} - (yz - 0)\hat{\jmath} + (0-0)\hat{k}$$

$$\nabla \times F = (-xz - 2z)\hat{\imath} - yz\hat{\jmath} + 0\hat{k}$$

$$(\vec{\nabla}\times\vec{F})\cdot\vec{n} = 0$$

$$\iint_S 0\, dA = 0$$

$$\int_0^{2\pi}\int_0^2 0 \cdot r\,dr\,d\theta = 0$$

$\ln x - \ln y$

14. Find both implicit partial derivatives of the function $x^5 - xyz + z^4 = \ln\left(\frac{x}{y}\right)$. (12 points)

$F(x,y,z) = x^5 - xyz + z^4 - \ln x + \ln y$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{5x^4 - yz - \frac{1}{x}}{xy + 4z^3}\left(\frac{x}{x}\right) = \frac{-5x^4 + xyz + 1}{x^2y + 4xz^3}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz + \frac{1}{y}}{xy + 4z^3}\left(\frac{y}{y}\right) = \frac{xzy - 1}{xy^2 + 4yz^3}$$

15. Find $\frac{dw}{dt}$ for $w = x\sin(y), x = e^t, y = \pi - t$. (8 points)

$$\frac{\partial w}{\partial x} = \sin y = \sin(\pi - t) \qquad \frac{dx}{dt} = e^t$$

$$\frac{\partial w}{\partial y} = x\cos y = e^t \cos(\pi - t) \qquad \frac{dy}{dt} = -1$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} = \sin(\pi - t) e^t + e^t \cos(\pi - t)(-1)$$

$$= e^t \left[\sin(\pi - t) - \cos(\pi - t) \right]$$