Math 2153, Exam #2, Summer 2015 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Use the second partials test to find the extrema of the function $f(x,y) = 2x^4 - 6x^2y + y^2$. Classify all the critical points as a minimum, maximum or a saddle point. If this is not possible with the second partials test, explain why not. (18 points)

$f_x = 8x^3 - 12xy = 0 \quad 4x(x^2 - 3y) = 0 \quad x = 0 \quad x^2 = 3y \quad x^2 = 3(3x^2)$

$x^2 = 9x^2$

$\Rightarrow x = 0$ again

$f_y = -6x^2 + 2y = 0 \quad 2y = 6x^2 \Rightarrow y = 3x^2$

$\Rightarrow y = 0$

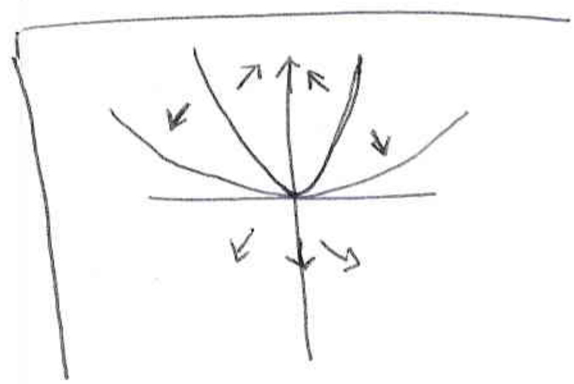
critical point $(x,y) = (0,0)$

$f_{xx} = 24x^2 - 12y \Rightarrow 0 \qquad (0)(2) - 0 = 0$

$f_{yy} = 2$

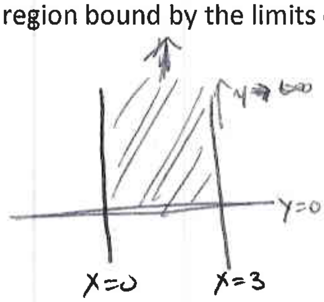
Cannot be determined by this test

$f_{xy} = -12x \Rightarrow 0$



2. Evaluate each integral. Sketch or describe the region bound by the limits of integration. (12 points each)

a. $\displaystyle\int_0^3\int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx$



$\int_0^3 x^2 \arctan y \Big|_0^\infty dx =$

$\int_0^3 x^2 \pi/2 \, dx = \quad x^3 \cdot \frac{\pi}{6}\Big|_0^3 = \frac{27\pi}{6} = \frac{9\pi}{2}$

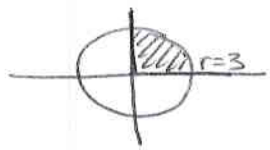
b. $\int_0^{\pi/2}\int_0^3 re^{-r^2}\,dr\,d\theta$

$u = r^2$
$du = 2r\,dr$
$-\frac{1}{2}du = r\,dr$

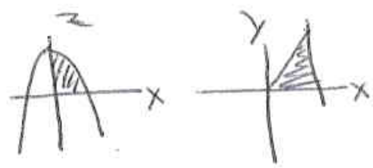
$\int -\frac{1}{2}e^u\,du = -\frac{1}{2}e^u$

$\int_0^{\pi/2} -\frac{1}{2}e^{-r^2}\Big|_0^3\,d\theta = -\frac{1}{2}\int_0^{\pi/2} e^{-9}-1\,d\theta = \frac{1}{2}(1-e^{-9})\cdot\pi/2$

$= \boxed{\frac{\pi}{4}(1-e^{-9})}$



c. $\int_0^3\int_0^x\int_0^{9-x^2} dz\,dy\,dx$



$\int_0^3\int_0^x z\Big|_0^{9-x^2} dy\,dx = \int_0^3\int_0^x 9-x^2\,dy\,dx$

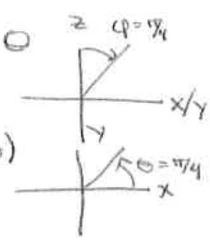
$= \int_0^3 (9-x^2)y\Big|_0^x\,dx = \int_0^3 9x - x^3\,dx = \frac{9}{2}x^2 - \frac{1}{4}x^4\Big|_0^3 =$

$\frac{9}{2}\cdot 9 - \frac{1}{4}\cdot 81 = \frac{81}{2} - \frac{81}{4} = \boxed{\frac{81}{4}}$

d. $\int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\cos\theta} \rho^2 \sin\varphi\cos\varphi\,d\rho\,d\theta\,d\varphi$

$\rho = \cos\Theta$

$\cos\Theta(1-\sin^2\Theta)$



$\int_0^{\pi/4}\int_0^{\pi/4} \frac{1}{3}\rho^3\Big|_0^{\cos\theta} \sin\varphi\cos\varphi\,d\Theta\,d\varphi$

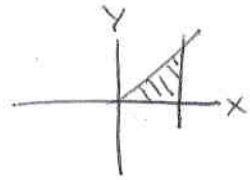
$\frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4} \cos^3\Theta\cdot\sin\varphi\cos\varphi\,d\Theta\,d\varphi = \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4}(\cos\Theta - \sin^2\Theta\cos\Theta)$
$\sin\varphi\cos\varphi\,d\Theta d\varphi$

$= \frac{1}{3}\int_0^{\pi/4}(\sin\Theta - \frac{1}{3}\sin^3\Theta\Big|_0^{\pi/4}\sin\varphi\cos\varphi\,d\varphi = \frac{1}{3}\int_0^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3\right)\sin\varphi\cos\varphi\,d\varphi$

$\frac{1}{3}\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)\frac{1}{2}\sin^2\varphi\Big|_0^{\pi/4} = \frac{1}{6}\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{12}\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) = \frac{5}{72\sqrt{2}}$

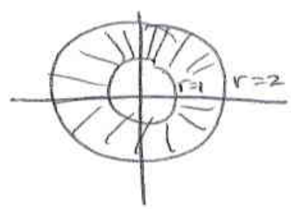
3. Set up a double integral to find the volume of the solid bounded by the graphs: $z = xy, z = 0, y = x, x = 1$. You don't need to integrate, just set it up. [Hint: a sketch of the region may help you.] (10 points)

$$\int_0^1 \int_0^x xy \, dy \, dx$$

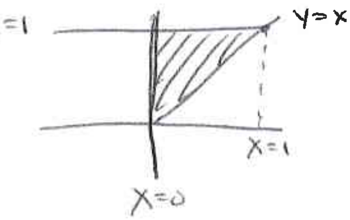


4. Set up a double integral to find the volume of the solid bounded by the graphs, in polar coordinates: $z = \ln\left(x^2 + y^2\right), z = 0, x^2 + y^2 \geq 1, x^2 + y^2 \leq 4$. You don't need to integrate it, just set it up so that it is easy to complete it. (15 points)

$$\int_0^{2\pi} \int_1^2 \ln(r^2)\, r \, dr \, d\theta$$



5. Change the order of integration and find the value of the integral $\int_0^1 \int_x^1 x\sqrt{1+2y^3}\,dy\,dx$. (12 points)



$$\int_0^1 \int_0^y x\sqrt{1+2y^3}\,dx\,dy =$$

$$\int_0^1 \frac{x^2}{2}\sqrt{1+2y^3}\Big|_0^y dy = \frac{1}{2}\int_0^1 y^2\sqrt{1+2y^3}\,dy$$

$u = 1+2y^3$

$du = 6y^2 dy$

$\frac{1}{6}du = y^2 dy$

$$\frac{1}{2}\int \frac{1}{6}u^{1/2}\,du = \frac{1}{\cancel{6}_3}\cdot\frac{\cancel{2}}{3}u^{3/2} = \frac{1}{18}u^{3/2}$$

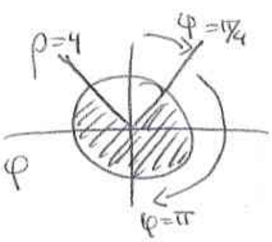
$$\frac{1}{18}(1+2y^3)^{3/2}\Big|_0^1 = \frac{1}{18}(3)^{3/2} - \frac{1}{9}(1)^{3/2} = \boxed{\frac{1}{18}(3^{3/2}-1)}$$

6. Find a triple integral to calculate the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cone $z=\sqrt{x^2+y^2}$. Find the value of the integral. [Hint: This will be easiest to evaluate in spherical coordinates.] (14 points)

$\rho = 4$ sphere

$\varphi = \pi/4$ cone

$$\int_{\pi/4}^{\pi}\int_0^{2\pi}\int_0^4 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$



$$\int_{\pi/4}^{\pi}\int_0^{2\pi}\frac{1}{3}\rho^3\Big|_0^4 \sin\varphi\,d\varphi\,d\theta = \frac{64}{3}\int_{\pi/4}^{\pi}\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi$$

$$= \frac{128\pi}{3}\int_{\pi/4}^{\pi}\sin\varphi\,d\varphi = \frac{128\pi}{3}(-\cos\varphi)\Big|_{\pi/4}^{\pi} = \frac{128\pi}{3}\left(-(-1)-\left(-\frac{1}{\sqrt{2}}\right)\right) =$$

$$\boxed{\frac{128\pi}{3}\left(1+\frac{1}{\sqrt{2}}\right)}$$

7. For each vector field below, determine if the field is conservative. If it is, find the potential function. If it is not, prove it. (13 points each)

a. $F(x,y) = 3x^2y^2\vec{i} + 3x^3y\vec{j}$

M N

$$\frac{\partial M}{\partial y} = 6x^2y \qquad \frac{\partial N}{\partial x} = 9x^2y$$

it is not conservative

$$\vec{\nabla} \times \vec{F} = \langle 0, 0, -3x^2y \rangle \neq \vec{0}$$

b. $F(x,y,z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^3\vec{k}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^3 \end{vmatrix} =$$

$$(6xyz^3 - 6xyz^2)\hat{i} - (3y^2z^3 - 3y^2z^2)\hat{j} + (2yz^3 - 2yz^3)\hat{k}$$

$$\neq \vec{0}$$

the field is not conservative