# Math 2415 Homework #7 Key



1. $\alpha^2 u_{xx} = u_t \qquad u(0,t)=10 \qquad u(50,t)=40$

$u(x,t) = v(x) + w(x,t) \qquad v(x) = \left(\frac{40-10}{50}\right)x + 10 = \frac{3}{5}x+10$

$w(0,t)=0, \quad w(50,t)=0$

$$u(x,t) = \frac{3}{5}x + 10 + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{50}\right) e^{-\frac{a^2 n^2 \pi^2}{2500}t}$$

The steady state solution is $\frac{3}{5}x+10$.

2. Since there are derivatives in the initial conditions, these will be cosine functions rather than sine functions.

You may assume $\alpha^2 = 1$.

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{900}t}$$

$$c_0 = \frac{2}{30}\int_0^{30} x\,dx = \frac{1}{15}\cdot\frac{1}{2}x^2\Big|_0^{30} = \frac{1}{30}(900-0) = 30$$

$$c_n = \frac{2}{30}\int_0^{30} x\cos\left(\frac{n\pi x}{30}\right)\Big|_0^{30}$$

| +/- | u | dv |
|---|---|---|
| + | $x$ | $\cos\left(\frac{n\pi x}{30}\right)$ |
| − | 1 | $\frac{30}{n\pi}\sin\left(\frac{n\pi x}{30}\right)$ |
| + | 0 | $-\frac{900}{n^2\pi^2}\cos\left(\frac{n\pi x}{30}\right)$ |

$$\frac{1}{15}\left[\cancel{\frac{30x}{n\pi}\sin\left(\frac{n\pi x}{30}\right)} + \frac{900}{n^2\pi^2}\cos\left(\frac{n\pi x}{30}\right)\right|_0^{30}$$

$$\frac{60}{n^2\pi^2}\left[(-1)^n - 1\right] \qquad \begin{array}{l} n \text{ even} = 0 \\ n \text{ odd} = -2 \end{array} \qquad c_n = \frac{-120}{(2k+1)^2\pi^2}$$

$$u(x,t) = 15 - \sum_{n=1}^{\infty} \frac{120}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{30}\right) e^{-\frac{(2k+1)^2\pi^2}{900}t}$$

3. a. $x_1 = u \quad x_2 = u' \quad x_1' = x_2 \qquad x_2' = u'' \qquad u'' = -\frac{1}{2}u' - 2u$

$x_1' = x_2$

$x_2' = -2x_1 - \frac{1}{2}x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3b. $t^2u'' = -tu' - (t^2 - \frac14)u$ $\quad x_1 = u \quad x_1' = x_2 = u' \quad x_2' = u''$

$$x_1' = x_2$$
$$x_2' = (\tfrac14 - t^2)x_1 - tx_2$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac14 - t^2 & -t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c. $u = x_1 \quad u' = x_2 = x_1' \quad x_2' = u'' = x_3 \quad x_3' = u''' = x_4 \quad x_4' = u^{iv} \quad u^{iv} = u$

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = x_4$$
$$x_4' = x_1$$
$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

d. $u'' = -\frac14 u' - 4u \quad x_1 = u, \; x_1' = x_2 = u' \quad x_2' = u''$

$$x_1' = x_2$$
$$x_2' = -4x_1 - \tfrac14 x_2 \quad + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -\frac14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\cos 3t \end{bmatrix}$$

$x_1(0) = 1, \; x_2(0) = -2$

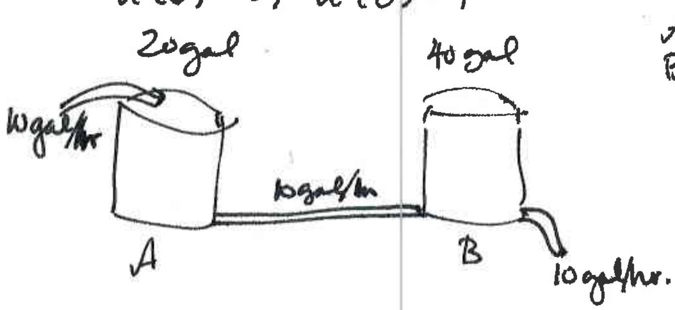
4 a. $x_1' - 3x_1 = -2x_2 \Rightarrow -\frac12 x_1' + \frac32 x_1 = x_2$

$$x_2' = -\tfrac12 x_1'' + \tfrac32 x_1' = 2x_1 - 2(-\tfrac12 x_1' + \tfrac32 x_1)$$
$$-\tfrac12 x_1'' + \tfrac32 x_1' = 2x_1 + x_1' - 3x_1$$
$$-\tfrac12 x_1'' + \tfrac12 x_1' + x_1 = 0 \quad x(-2)$$
$$x_1'' - x_1' - 2x_1 = 0 \Rightarrow u'' - u' - 2u = 0 \quad u(0) = 3, \; u'(0) = \tfrac12$$

b. $\frac12 x_1' = x_2 \quad x_2' = \frac12 x_1'' = -2x_1 \quad x_1'' + 4x_1 = 0 \Rightarrow u'' + 4u = 0$

$u(0) = 3, \; u'(0) = 4$

5.



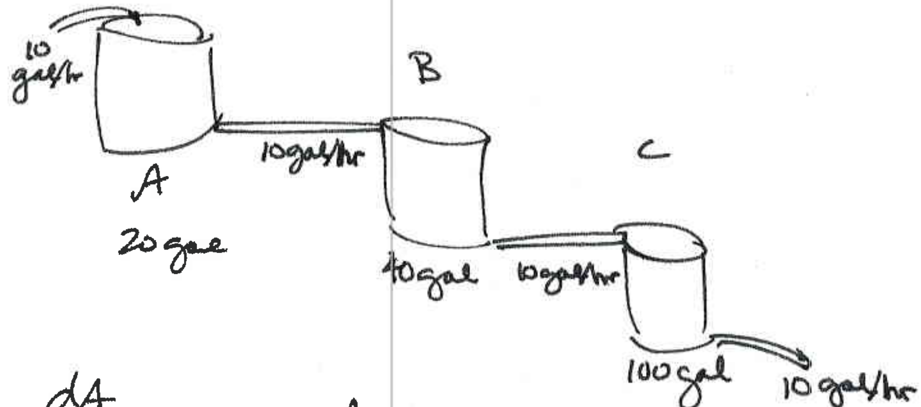

$A(0) = 500$
$B(0) = 500$

$$\frac{dA}{dt} = 0 \cdot \frac{10\,gal}{hr} - \frac{A}{20g} \cdot \frac{10\,gal}{hr}$$

$$\frac{dB}{dt} = \frac{A}{20g} \cdot \frac{10\,gal}{hr} - \frac{B}{40} \cdot \frac{10\,gal}{hr}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}' = \begin{bmatrix} -\frac12 & 0 \\ \frac12 & -\frac14 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

6.




$A(0) = B(0) = C(0) = 500$

$$\frac{dA}{dt} = 0 - \frac{A}{20}\cdot 10$$

$$\frac{dB}{dt} = \frac{A}{20}\cdot 10 - \frac{B}{40}\cdot 10$$

$$\frac{dC}{dt} = \frac{B}{40}\cdot 10 - \frac{C}{100}\cdot 10$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}' = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

7. $AB = \begin{bmatrix} 16 & 9 & -6 \\ 22 & 19 & -12 \\ 16 & 4 & -7 \end{bmatrix}$ $\quad BA = \begin{bmatrix} 5 & 6 & 4 \\ 5 & -6 & -3 \\ -4 & 16 & 29 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 6/7 & -3/7 & -1/7 \\ 3/2 & -1/2 & -1 \\ -4/7 & 2/7 & 3/7 \end{bmatrix}$$

8. $(1-2i)(2) + (i)(3+i) + 2(1-2i) = 2 - 4i + 3i - 1 + 2 - 4i = 3 + 5i$

9. $x' = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$ $\quad \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3(4) - 2(2) \\ 2(4) - 2(2) \end{pmatrix} = \begin{pmatrix} 12-4 \\ 8-4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$

10. $x' = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}$ $\quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 6e^{-t} \\ -8e^{-t} + 2e^{2t} \\ -4e^{-t} - 2e^{2t} \end{pmatrix} =$

$$= \begin{pmatrix} 6e^{-t} - 8e^{-t} + 2e^{2t} - 4e^{-t} - 2e^{2t} \\ 12e^{-t} - 8e^{-t} + 2e^{2t} + 4e^{-t} + 2e^{2t} \\ 0 + 8e^{-t} - 2e^{2t} - 4e^{-t} - 2e^{2t} \end{pmatrix} = \begin{pmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t} \\ 4e^{-t} - 4e^{2t} \end{pmatrix}$$

11. $\Psi' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} = \begin{pmatrix} e^{-3t} - 4e^{-3t} & e^{2t} + e^{2t} \\ 4e^{-3t} + 8e^{-3t} & 4e^{2t} - 2e^{2t} \end{pmatrix}$

$$= \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$$