Math 2415 Homework #6

1.a. $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = 1\int_0^1 1-x\,dx + \int_{-1}^{0} x+1\,dx = x-\frac{1}{2}x^2\Big|_0^1 + \frac{1}{2}x^2+x\Big|_{-1}^{0}$

$= 1-0-\frac{1}{2}(1)+0+0+0-\frac{1}{2}(1)+1 = 2-1 = 1$

$a_n = \int_{-1}^{0} (1+x)\cos(n\pi x)\,dx + \int_0^1 (x-1)\cos(n\pi x)\,dx =$

u	dv
$1-x$	$\cos(n\pi x)$
-1	$\sin(n\pi x)\cdot\frac{1}{n\pi}$
0	$-\cos(n\pi x)\cdot\frac{1}{n^2\pi^2}$

u	dv
$x+1$	$\cos(n\pi x)$
1	$\sin(n\pi)\cdot\frac{1}{n\pi}$
0	$-\cos(n\pi x)\frac{1}{n^2\pi^2}$

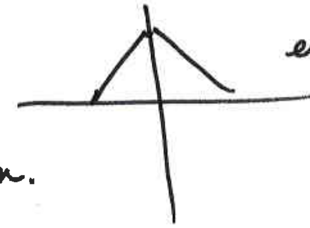
$\frac{(1+x)}{n\pi}\sin(n\pi x)+\frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_{-1}^{0} +$

$\frac{(x-1)}{n\pi}\sin(n\pi x) - \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_0^1$

$= \frac{1}{n^2\pi^2}\left[1-(-1)^n\right] + \left[\frac{-1}{n^2\pi^2}\right]\left[(-1)^n-1\right] = \frac{1}{n^2\pi^2}\left[1-(-1)^n+1-(-1)^n\right] =$

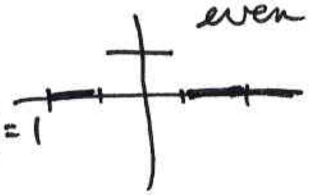
$\frac{2}{n^2\pi^2}\left[1-(-1)^n\right]$ when n is even, $n=2k$, $1-(-1)^n=0$
when n is odd, $n=2k+1$, $1-(-1)^n = 2$

$= \frac{4}{(2k+1)^2\pi^2}$


even

$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = 0$ since function is even.

$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}\cos((2k+1)\pi x)$

even


b. $a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{\pi}x\Big|_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left[\pi/2-(-\pi/2)\right] = \frac{1}{\pi}[\pi] = 1$

$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\cdot\frac{1}{n}\sin\left(\frac{n\pi x}{\pi}\right)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right]$

$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$ when $n=2k$ $\sin\left(\frac{2k\pi}{2}\right)=0$
when $n=2k+1$ $\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$

$= \frac{2}{(2k+1)\pi}(-1)^k$ So since $f(x)$ is even $b_n = 0$

16. cont'd.

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k+1)\pi} \cos((2k+1)x)$$

c. $a_0 = \frac{1}{1}\int_{-1}^{1} 1-x^2\,dx = 2\int_0^1 1-x^2\,dx = 2\left(x-\frac{1}{3}x^3\right)\Big|_0^1 =$

$2\left(1-\frac{1}{3}-0\right) = \frac{4}{3}$

even

$a_n = \int_{-1}^{1} (1-x^2)\cos(n\pi x)\,dx$

	u	dv
+	$1-x^2$	$\cos(n\pi x)$
−	$-2x$	$\frac{1}{n\pi}\sin(n\pi x)$
+	-2	$\frac{-1}{n^2\pi^2}\cos(n\pi x)$
−	0	$\frac{-1}{n^3\pi^3}\sin(n\pi x)$

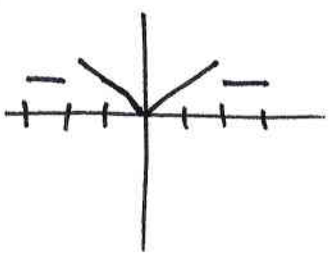
$= 2\left[\cancel{\frac{(1-x^2)}{n\pi}\sin(n\pi x)} - \frac{2x}{n^2\pi^2}\cos(n\pi x) + \cancel{\frac{2}{n^3\pi^3}\sin(n\pi x)}\right]_0^1$

$= 2\left[0 - \frac{2}{n^2\pi^2}\cos(n\pi) + 0\right] = -\frac{4}{n^2\pi^2}\cos(n\pi) = \frac{-4(-1)^n}{n^2\pi^2}$

$b_n = 0$ since $f(x)$ is even.

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2\pi^2}\cos(n\pi x)$$

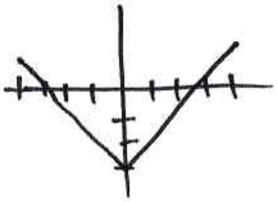
2a.



$$f(x) = \begin{cases} 1 & -3 \le x < 2 \\ -x & -2 \le x < 0 \\ x & 0 \le x < 2 \\ 1 & 2 \le x < 3 \end{cases}$$

period = 6

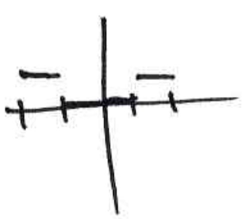
b.



$$f(x) = \begin{cases} -x-3 & -4 \le x \le 0 \\ x-3 & 0 \le x \le 4 \end{cases}$$

period = 8

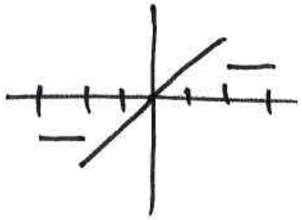
c.



$$f(x) = \begin{cases} 1 & -2 \le x < -1 \\ 0 & -1 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$$

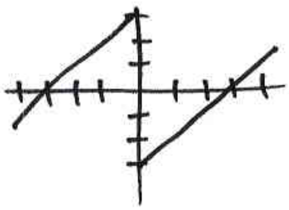
period = 4

3a.



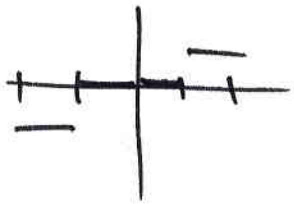
$$f(x) = \begin{cases} -1 & -3 \le x < -2 \\ x & -2 \le x < 2 \\ 1 & 2 \le x < 3 \end{cases} \qquad \text{period} = 6$$

b.



$$f(x) = \begin{cases} x+3 & -4 \le x < 0 \\ x-3 & 0 \le x < 4 \end{cases} \qquad \text{period} = 8$$

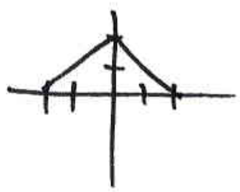
c.



$$f(x) = \begin{cases} -1 & -2 \le x < -1 \\ 0 & -1 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases} \qquad \text{period} = 4$$

4. $f(x) = 2-x, \quad 0 \le x < 2$

a.



even
$b_n = 0$

$$f(x) = \begin{cases} 2+x & -2 \le x < 0 \\ 2-x & 0 \le x < 2 \end{cases}$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\cdot\frac{1}{2}(2)(4) = 2$$

$$a_n = \frac{1}{2}\int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{-2}^{0}(2+x)\cos\left(\frac{n\pi x}{2}\right)dx$$

	u	dv
$+$	$2-x$	$\cos\left(\frac{n\pi x}{2}\right)$
$-$	-1	$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$
$+$	0	$\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$

	u	dv
$+$	$2+x$	$\cos\left(\frac{n\pi x}{2}\right)$
$-$	1	$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$
$+$	0	$\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$

$$\cancel{\frac{1}{2}}\left[\cancel{\frac{2(2-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right)} - \frac{\cancel{4}^{\,2}}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \cancel{\frac{1}{2}}\left[\cancel{\frac{2(2+x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right)} + \frac{\cancel{4}^{\,2}}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0}$$

$$\frac{-2}{n^2\pi^2}\left[\cos(n\pi)-1\right] + \frac{2}{n^2\pi^2}\left[1-\cos(n\pi)\right] = \frac{2}{n^2\pi^2}\left[1-\cos(n\pi)+1-\cos(n\pi)\right]$$

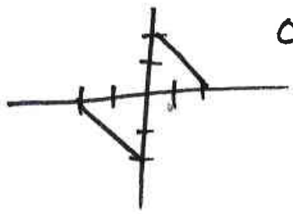
$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

when $n = 2k$ $\quad 1-(-1)^n = 0$
when $n = 2k+1$ $\quad 1-(-1)^n = 2$

4a cont'd.

$$a_k = \frac{8}{(2k+1)^2\pi^2} \qquad f(x) = 1 + \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2\pi^2} \cos\left(\frac{(2k+1)\pi x}{2}\right)$$

b.



odd $\quad a_n = 0$

$a_0 = 0$

$$f(x) = \begin{cases} -x-2 & -2 \le x < 0 \\ 2-x & 0 \le x < 2 \end{cases}$$

$$b_n = -\frac{1}{2}\int_{-2}^{0} (x+2)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} (2-x)\sin\left(\frac{n\pi x}{2}\right)dx =$$

	u	dv
$+$	$x+2$	$\sin\left(\frac{n\pi x}{2}\right)$
$-$	1	$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$
$+$	0	$-\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)$

	u	dv
$+$	$2-x$	$\sin\left(\frac{n\pi x}{2}\right)$
$-$	-1	$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$
$+$	0	$-\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)$

$$+\frac{1}{\cancel{2}}\left[+\frac{\cancel{2}(x+2)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \cancel{\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)}\right]_{-2}^{0} + \frac{1}{\cancel{2}}\left[\frac{-\cancel{2}(2-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right.$$

$$\left.\cancel{-\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)}\right|_0^2 = \frac{2}{n\pi}\cos(0) - 0 + 0 + \frac{2}{n\pi}\cos(0) = \frac{4}{n\pi}$$

$$f(x) = \sum_{k=1}^{\infty} \frac{4}{k\pi}\sin\left(\frac{k\pi x}{2}\right)$$

See attached graphs

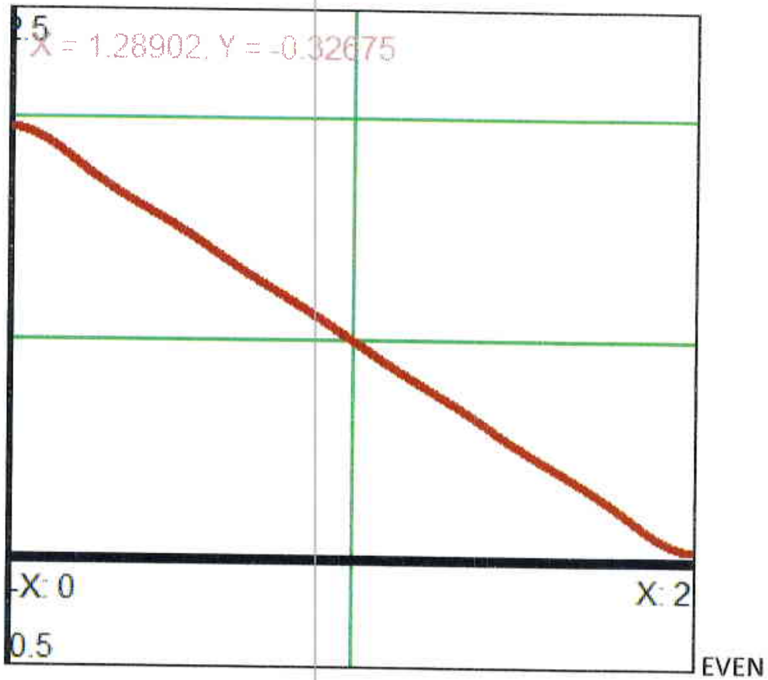
c. it looks like the even graph did a better job in 5 terms. the coefficients converge faster.

5a. $u = XT \qquad u_{xx} = X''T \qquad u_t = XT'$

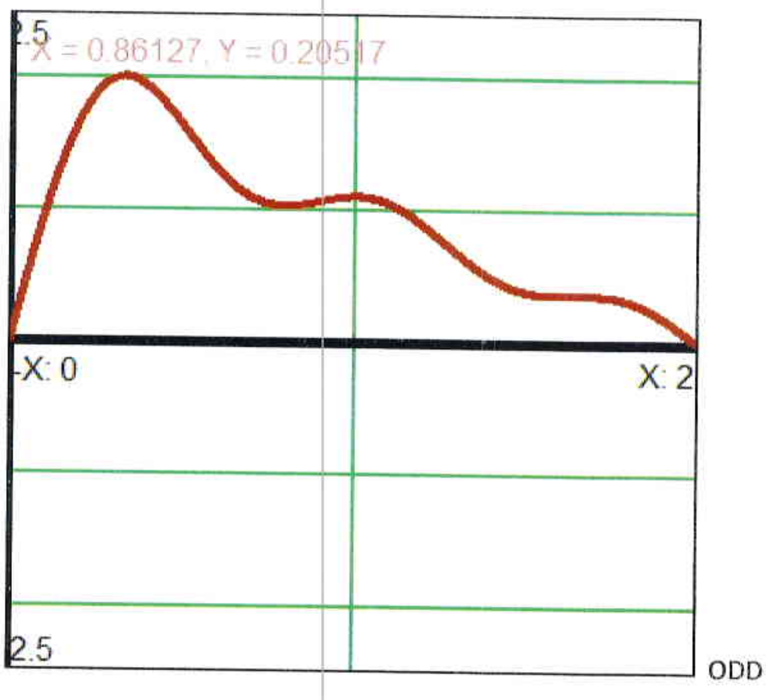
$$xX''T = -XT' \Rightarrow \frac{xX''}{X} = -\frac{T'}{T} \quad \text{Separation works.}$$

b. $tX''T = -xXT'$

$$\Rightarrow \frac{X''}{xX} = \frac{-T'}{tT} \quad \text{Separation works}$$


2.5
X = 1.28902, Y = -0.32675
X: 0
X: 2
0.5
EVEN

4a


2.5
X = 0.86127, Y = 0.20517
X: 0
X: 2
2.5
ODD

4b

5c. $X''Y = -(x+y)XY''$ Cannot be separated

d. $X''Y + XY'' + xXY = 0$ $\quad X''Y + xXY = -XY''$

$$Y(X'' + xX) = -XY''$$

$$\Rightarrow \frac{X'' + xX}{X} = -\frac{Y''}{Y}$$ Separation works

6. $100\, u_{xx} = u_t$ $\quad L = 1$ $\quad u(x,0) = \sin(2\pi x) - \sin(5\pi x)$

$\alpha^2 = 100$

$$u(x,t) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi x}{L}\right) e^{-\frac{\alpha^2 k^2 \pi^2}{L^2} t} = \sum_{k=1}^{\infty} C_k \sin(k\pi x) e^{-100k^2\pi^2 t}$$

$$C_n = \frac{2}{L}\int_0^1 [\sin(2\pi x) - \sin(5\pi x)] \sin(k\pi x)\, dx \Rightarrow 0 \text{ unless } k = 2, 5 \text{ by}$$

orthogonality of $\sin(nx)$ & $\sin(mx)$

$$= \frac{2}{2}\int_0^1 \underbrace{\sin^2(2\pi x)}_{C_2} - \underbrace{\sin^2(5\pi x)}_{C_5}\, dx = 1\int_0^1 \underbrace{1 - \cos(4\pi x)}_{C_2} - \int_0^1 \underbrace{1 - \cos(10\pi x)}_{C_5}\, dx$$

$$= \underbrace{1 - \frac{1}{4\pi}\sin(4\pi x)\Big|_0^1}_{C_2} - \underbrace{\left[1 - \frac{1}{10\pi}\sin(10\pi x)\right]\Big|_0^1}_{C_5} \Rightarrow 1 = C_2 \quad -1 = C_5$$

$$u(x,t) = \left(\sin(2\pi x) - \sin(5\pi x)\right) e^{-100k^2\pi^2 t}$$

7. $L = 40$ $\quad u(0,t) = 0,\ u(40,t) = 0$ $\quad u(x,0) = \begin{cases} x & 0 \le x \le 20 \\ 40 - x & 20 \le x < 40 \end{cases}$

$$u(x,t) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi x}{40}\right) e^{-\frac{k^2\pi^2}{L^2} t}$$

$$C_n = \frac{2}{40}\int_0^{20} x \sin\left(\frac{n\pi x}{40}\right) dx + \frac{1}{20}\int_{20}^{40} (40 - x)\sin\left(\frac{n\pi x}{40}\right) dx$$

	u	dv
+	x	$\sin\left(\frac{n\pi x}{40}\right)$
−	1	$-\frac{40}{n\pi}\cos\left(\frac{n\pi x}{40}\right)$
+	0	$-\frac{1600}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right)$

	u	dv
+	$40 - x$	$\sin\left(\frac{n\pi x}{40}\right)$
−	-1	$\frac{-40}{n\pi}\cos\left(\frac{n\pi x}{40}\right)$
+	0	$\frac{-1600}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right)$

7 cont'd.

$$\frac{1}{\cancel{20}}\left[-\frac{\overset{2}{\cancel{40}}x}{n\pi}\cos\left(\frac{n\pi x}{40}\right)+\frac{\overset{80}{\cancel{1600}}}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right)\right|_0^{20}+\frac{1}{\cancel{20}}\left[\frac{-(40-x)\overset{2}{\cancel{40}}}{n\pi}\cos\left(\frac{n\pi x}{40}\right)-\frac{\overset{80}{\cancel{1600}}}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right)\right|_{20}^{40}$$

$$\cancel{\frac{-2(20)}{n\pi}\cos\left(\frac{n\pi\cdot 20}{40}\right)}+\frac{80}{n^2\pi^2}\sin\left(\frac{n\pi\cdot 20}{40}\right)+\cancel{\frac{2\cdot 0}{n\pi}\cos(0)}-\cancel{\frac{80}{n^2\pi^2}(0)}+\cancel{\frac{-2(0)}{n\pi}\cos(n\pi)}-\cancel{\frac{80}{n^2\pi^2}\sin(n\pi)}$$

$$+\cancel{\frac{2(20)}{n\pi}\cos\left(\frac{n\pi\cdot 20}{40}\right)}+\frac{80}{n^2\pi^2}\sin\left(\frac{n\pi\cdot 20}{40}\right)=\frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

n is even
$n=2k$
$\sin\left(\frac{2k\pi}{2}\right)=0$

when n is odd $\quad n=2k+1 \quad \sin\left(\frac{(2k+1)\pi}{2}\right)=(-1)^k$

$$c_k=\frac{160(-1)^k}{(2k+1)^2\pi^2}$$

$$u(x,t)=\sum_{k=0}^{\infty}\frac{160(-1)^k}{(2k+1)^2\pi^2}\sin\left(\frac{(2k+1)\pi x}{40}\right)e^{-\frac{(2k+1)^2\pi^2}{1600}t}$$