2415 Homework #3 key

1. a. $6y'' - 5y' + y = 0 \quad y(0) = 4,\ y'(0) = 0 \qquad y = e^{rt}$

$6r^2 - 5r + 1 = 0 \qquad (3r-1)(2r-1) = 0 \quad r = 1/3,\ r = 1/2$

$y(t) = c_1 e^{1/3 t} + c_2 e^{1/2 t} \qquad 4 = c_1 + c_2 \qquad 4 = c_1 + c_2$

$y'(t) = \frac{1}{3} c_1 e^{1/3 t} + \frac{1}{2} c_2 e^{1/2 t} \quad (0 = \frac{1}{3} c_1 + \frac{1}{2} c_2) \cdot -2 \Rightarrow \quad 0 = -\frac{2}{3} c_1 - c_2$

$4 = 1/3\, c_1 \Rightarrow c_1 = 12$

$y(t) = 12 e^{1/3 t} - 8 e^{1/2 t}$ critical point at $x = 0$ $c_2 = -8$

(note $y'(0) = 0$ is given).

function goes to $-\infty$ as $t \to \infty$

b. $2y'' - 3y' + y = 0 \qquad y(0) = 2,\ y'(0) = \frac{1}{2} \qquad y = e^{rt}$

$2r^2 - 3r + 1 = 0 \qquad (2r-1)(r-1) = 0 \qquad r = 1/2,\ r = 1$

$y(t) = c_1 e^{1/2 t} + c_2 e^{t} \qquad 2 = c_1 + c_2$

$y'(t) = 1/2\, c_1 e^{1/2 t} + c_2 e^{t} \quad -\frac{1}{2} = -\frac{1}{2} c_1 - c_2 \longrightarrow \quad \frac{3}{2} = \frac{1}{2} c_1 \Rightarrow c_1 = 3$

$c_2 = -1$

$y(t) = 3 e^{1/2 t} - e^{t}$ critical point at $\approx t = .8109$

function $\to -\infty$ as $t \to \infty$

c. $y'' + 4y' + 5y = 0 \qquad y(0) = 1,\ y'(0) = 0 \qquad y = e^{rt}$

$r^2 + 4r + 5 = 0 \qquad r = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t \qquad e^{(-2 \pm i)t} = e^{-2t} \cos t,\ e^{-2t} \sin t$

$y'(t) = -2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t$

$1 = c_1 \qquad 0 = -2c_1 + c_2 \qquad c_2 = 2$

$y(t) = e^{-2t} \cos t + 2e^{-2t} \sin t$ decays to 0 as $t \to \infty$

multiple critical points

first > 0 at $t = 0$, $t = 3.14$ (π) is first min.
max.

d. $y'' + 4y' + 4y = 0 \qquad y(-1) = 2 \qquad y'(-1) = 1 \qquad y = e^{rt}$

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \quad r = 2$ repeated

$y(t) = c_1 e^{2t} + c_2 t e^{2t} \qquad 2 = c_1 e^{-2} - c_2 e^{-2} \Rightarrow 2e^2 = c_1 - c_2$

$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} \qquad 1 = 2c_1 e^{-2} + c_2 e^{-2} - 2c_2 e^{-2}$

$y(t) = -e^{2t+2} + 3te^{2t+2} \qquad e^2 = 2c_1 - c_2$

$-2e^2 = -c_1 + c_2$

min at $t = -1/6$ goes to ∞ as $t \to \infty$ $\qquad -e^2 = c_1 \qquad c_2 = 3e^2$

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2a. $y'' + 2y = 0 \quad y'(0) = 4, \quad y'(\pi) = 0 \qquad y = e^{rt}$

$r^2 + 2 = 0 \qquad r = \pm\sqrt{2}i \qquad y(t) = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t$

$y'(t) = -\sqrt{2}c_1 \sin\sqrt{2}t + \sqrt{2}c_2 \cos\sqrt{2}t \qquad 4 = \cancel{-\sqrt{2}c_1(0)} + \sqrt{2}c_2$

$c_2 = 2\sqrt{2}$

$0 = -\sqrt{2}c_1 \sin\sqrt{2}\pi + 4 \quad \cos\sqrt{2}\pi$

$4\cos(\sqrt{2}\pi) = \sqrt{2}\sin(\sqrt{2}\pi)\cdot c_1 \Rightarrow c_1 = \frac{4}{\sqrt{2}}\tan(\sqrt{2}\pi) = c_1 = 2\sqrt{2}\tan\sqrt{2}\pi$

$y(t) = [2\sqrt{2}\tan(\sqrt{2}\pi)]\cos(\sqrt{2}t) + 2\sqrt{2}\sin(\sqrt{2}t)$

critical point at π and other points, oscillates forever.

b. $y'' + y = 0 \quad y(0) = 0, \quad y(L) = 0 \qquad y = e^{rt}$

$r^2 + 1 = 0 \qquad r = \pm i \qquad y(t) = c_1\cos t + c_2 \sin t$

$0 = c_1 \qquad 0 = c_2 \sin L \Rightarrow c_2 = 0 \qquad y(t) = 0$ trivial solution

no critical pts.

c. $x^2y'' + 3xy' + y = 0 \qquad y(1) = 0, \; y(e) = 0 \qquad y = t^n$

$n(n-1) + 3n + 1 = 0 \Rightarrow n^2 - n + 3n + 1 \Rightarrow n^2 + 2n + 1 = 0 \Rightarrow n = 1$ repeated

$y(t) = c_1 t + c_2 t\ln t \qquad 0 = c_1 + 0 \Rightarrow c_1 = 0$

$0 = c_2 e \Rightarrow c_2 = 0$ trivial solution

no critical points

3a. $\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$ is a fundamental set

b. $\begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x \neq 0$ is a fundamental set

$(-\infty, 0) \cup (0, \infty)$

c. $\begin{vmatrix} e^t\sin t & e^t\cos t \\ e^t\sin t + e^t\cos t & e^t\cos t - e^t\sin t \end{vmatrix} = \cancel{e^{2t}\sin t\cos t} - e^{2t}\sin^2 t - \cancel{e^{2t}\sin t\cos t} - e^{2t}\cos^2 t$

$= -e^{2t}(\sin^2 t + \cos^2 t) = -e^{2t} \neq 0$

is a fundamental set. exists everywhere

d. $\begin{vmatrix} t^2 & t^2\ln t & t^{-4} \\ 2t & 2t\ln t + t & -4t^{-5} \\ 2 & 2\ln t + 3 & 20t^{-6} \end{vmatrix} = t^2[(2t\ln t + t)(20t^{-6}) + (4t^{-5})(2\ln t + 3)] - t^2\ln t[40t^{-5}$

$+ 8t^{-5}] + t^{-4}[2t(2\ln t + 3) - (4t\ln t + 2t)] =$

$t^2[40t^{-5}\ln t + 20t^{-5} + 8t^{-5}\ln t + 12t^{-5}] - t^2\ln t[48t^{-5}] + t^{-4}[4t\ln t + 6t - 4t\ln t - 2t]$

3d cont'd

③

$t^2[48t^{-5}\ln t + 32t^{-5}] - t^2\ln t[48t^{-5}] + t^{-4}[4t] =$

$\cancel{48t^{-3}\ln t} + 32t^{-3} - \cancel{48t^{-3}\ln t} + 4t^{-3} = 36t^{-3} \neq 0$ is a fundamental set

B. $\begin{vmatrix} \sinh t & \cosh t & e^t \\ \cosh t & \sinh t & e^t \\ \sinh t & \cosh t & e^t \end{vmatrix} = \sinh t\,[e^t\sinh t - e^t\cosh t] - \cosh t\,[e^t\cosh t - e^t\sinh t] + e^t[\cosh^2 t - \sinh^2 t]$

$= \cancel{e^t\sinh^2 t} - e^t\sinh t\cosh t - e^t\cosh^2 t + \cancel{e^t\cosh t\sinh t} + e^t\cancel{\cosh^2 t} - \cancel{e^t\sinh^2 t} = 0$ not a fundamental set.

4a. $y'' + \frac{3}{t}y = 1 \qquad p(t) = 0 \qquad W = e^{-\int p(t)\,dt} = \text{constant}$

b. $y'' - \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)} \qquad p(t) = \frac{-3}{t-4} \qquad W = e^{-\int\frac{-3}{t-4}dt} = e^{3\ln|t-4|}$

$= (t-4)^3 \neq 0$

$(-\infty, 4) \cup (4, \infty)$

c. $y'' - \frac{x}{1-x\cot x}y' + \frac{y}{1-x\cot x} = 0 \qquad y(\pi/2) = 1 \quad y'(\pi/2) = 0$

$p(t) = \frac{x}{1-x\cot x} \qquad W = e^{-\int\frac{x}{1-x\cot x}\frac{\cos x}{\cos x}} = e^{\int\frac{x\cos x}{x\sin x - \cos x}}$

This is non-zero which is all we care about.
Though it's not defined when $\cot x$ has x = multiples of π & when $1 = x\cot x$

5a. $e^{1+2i} = e\cos 2t + ie\sin 2t$

b. $e^{(\ln 2)(1-i)} = e^{\ln 2}\cos(\ln 2) - ie^{\ln 2}\sin(\ln 2) = 2\cos(\ln 2) - 2i\sin(\ln 2)$

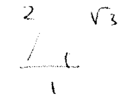
c. $e^{2-\pi/2 i} = e^2\cos\pi/2 - ie^2\sin\pi/2 = -e^2 i$

6a. $1+i = z \quad \|z\| = \sqrt{1^2+1^2} = \sqrt{2} = R \qquad \tan^{-1}(\frac{1}{1}) = \pi/4 \quad$ (1st Quad)

$= \sqrt{2}e^{\pi/4 i} = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$

6b. $\sqrt{3} - i = z \quad \|z\| = \sqrt{3+1} = 2 = R \qquad \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

Quad 4



$\sqrt{3} - i = 2(\cos \pi/6 - i \sin \pi/6) = 2e^{-\pi/6 i}$

7a. $x^3 = 1 \Rightarrow x^3 - 1 = 0 \quad (x-1)(x^2+x+1) = 0$ (small ones can be done by factoring)

$x = 1$ real principal root

$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Or $1 = e^{0i} = e^{2\pi i} = e^{4\pi i}$

$\sqrt[3]{1} = e^{\frac{0i}{3}} = 1; \quad e^{\frac{2\pi i}{3}} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} i;$

$e^{\frac{4\pi i}{3}} = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

b. $(1-i)^{1/2} \Rightarrow 1 - i = z = \sqrt{2}\, e^{-\pi/4} = \sqrt{2}\, e^{7\pi/4} = \sqrt{2}\, e^{15\pi/4 i}$

$\sqrt{(1-i)} \Rightarrow \left(\sqrt{2}\, e^{7\pi/4 i}\right)^{1/2} = \sqrt[4]{2}\, e^{7\pi/8 i} = \sqrt[4]{2}\left(\cos(7\pi/8) + i \sin(7\pi/8)\right)$

$\left(\sqrt{2}\, e^{15\pi/4 i}\right)^{1/2} = \sqrt[4]{2}\, e^{15\pi/8 i} = \sqrt[4]{2}\left(\cos(15\pi/8) + i \sin(15\pi/8)\right)$

8a. $x^2 y'' + x y' + y = 0 \qquad y = x^n$

$n(n-1) + n + 1 = 0 \Rightarrow n^2 - n + n + 1 = 0 \Rightarrow n^2 + 1 = 0 \quad n = \pm i$

$x^i = e^{(\ln x) i} = \cos[(\ln x)] + i \sin[(\ln x)]$

$y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

b. $t^2 y'' - t y' + 5y = 0 \qquad y = t^n$

$n(n-1) - n + 5 = 0 \Rightarrow n^2 - n - n + 5 = 0 \Rightarrow n - 2n + 5 = 0$

$n = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i \qquad t^{(1 \pm 2i)} = e^{\ln t} e^{(2 \ln t) i}$

$y(t) = c_1 t \cos[2 \ln t] + c_2 t \sin[2 \ln t]$

c. $t^2 y'' + 5t y' + 13 y = 0 \Rightarrow y = t^n \Rightarrow n(n-1) + 5n + 13 = 0$

$n^2 - n + 5n + 13 = 0 \Rightarrow n^2 + 4n + 13 = 0 \qquad n = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} =$

$-2 \pm 3i$

$t^{(-2+3i)} = e^{-2 \ln t} e^{3 \ln t} \qquad y(t) = c_1 t^{-2} \cos(3 \ln t) + c_2 t^{-2} \sin(3 \ln t)$