2415 Homework #2 Key

1a. $y' = \dfrac{x^2}{y(1+x^3)^4} \Rightarrow y\,dy = \dfrac{x^2\,dx}{(1+x^3)^4}$ $\qquad u = 1+x^3$, $\quad \frac{1}{3}du = \cancel{3}x^2 dx$

$\int y\,dy = \int \frac{1}{3}u^{-4}du \Rightarrow \frac{1}{2}y^2 = -\frac{1}{9}u^{-3} + C \Rightarrow y^2 = \dfrac{-2}{9(1+x^3)^3} + C$

b. $y' + y^2 \sin x = 0 \Rightarrow y' = -y^2 \sin x \Rightarrow \int \dfrac{dy}{y^2} = \int -\sin x\,dx$

$-\dfrac{1}{y} = \cos x + C \Rightarrow y = \dfrac{-1}{\cos x + C}$

c. $y' = (\cos^2 x)(\cos^2 2y) \Rightarrow \int \sec^2 2y\,dy = \int \cos^2 x\,dx$

$\qquad \frac{1}{2}(1+\cos 2x)$

$\cancel{\frac{1}{2}}\tan 2y = \cancel{\frac{1}{2}}(x + \frac{1}{2}\sin 2x) + C$

$\tan 2y = x + \frac{1}{2}\sin 2x + C$

d. $xy' = (1-y^2)^{1/2} \Rightarrow \int \dfrac{dy}{\sqrt{1-y^2}} = \int \dfrac{dx}{x} \Rightarrow \arcsin y = \ln x + C$

e. $y' = \dfrac{1-2x}{y}$ $\quad y(1) = -2 \Rightarrow \int y\,dy = \int (1-2x)dx \Rightarrow \frac{1}{2}y^2 = x - x^2 + C$

$y^2 = 2x - 2x^2 + C \Rightarrow 4 = 2-2+C \Rightarrow C=4$

$y = -\sqrt{2x-2x^2+4}$

f. $\sin 2x\,dx + \cos 3y\,dy = 0 \quad y(\pi/2) = \pi/3 \Rightarrow \int -\sin 2x\,dx = \int \cos 3y\,dy$

$\frac{1}{2}\cos 2x + C = \frac{1}{3}\sin 3y \Rightarrow \frac{1}{2}\cos(\pi) + C = \cancel{\frac{1}{3}\sin\left(\frac{3\pi}{3}\right)}^{\,0}$

$-\frac{1}{2} + C = 0 \Rightarrow C = \frac{1}{2}$

$\frac{1}{2}\cos 2x + \frac{1}{2} = \frac{1}{3}\sin 3y$

g. $y' = ty(4-y)$, $y(0) = y_0 \Rightarrow \dfrac{dy}{y(4-y)} = t\,dt \qquad \dfrac{1}{y(4-y)} = \dfrac{A}{y} + \dfrac{B}{4-y}$

$A(4-y) + By = 1 \quad y=0 \Rightarrow A4 = 1 \Rightarrow A\ 1/4;\ y = 4 \Rightarrow 4B = 1 \Rightarrow B = 1/4$

$\frac{1}{4}\int \frac{1}{y} - \frac{1}{y-4}\,dy = \int t\,dt \Rightarrow \frac{1}{4}(\ln y - \ln(y-4)) = \frac{1}{2}t^2 + C \Rightarrow \ln\left(\frac{y}{y-4}\right) = 2t^2 + C$

$\dfrac{y}{y-4} = e^{2t^2+C}$

2. $S(0) = 8000$ $S(3) = 0$ $r = .10$

$\frac{dS}{dt} = Sr + k = r(S + \frac{k}{r}) \Rightarrow \frac{dS}{S + \frac{k}{r}} = r\,dt$ $\ln|S + \frac{k}{r}| = rt + c$

$S + \frac{k}{r} = e^{rt+c} = S_0 e^{rt} \Rightarrow S(t) = S_0 e^{rt} - \frac{k}{r} \Rightarrow S(t) = S_0 e^{.1t} - 10k$

$t=0 \Rightarrow 8000 = S_0 - 10k$

$t=3 \Rightarrow 0 = S_0 e^{.3} - 10k \Rightarrow 10k = S_0 e^{.3}$

$8000 = S_0 - S_0 e^{.3} \Rightarrow S_0(1 - e^{.3}) = 8000 \Rightarrow S_0 = -22866.37$

$k = \frac{-22866.37}{10} e^{.3} = -3086.64$ would need to make payments of \$3086.64 per year.

pays \$9259.91 − \$8000 borrowed = \$1259.91 in interest.

3. $\frac{dy}{dt} = \frac{(.5 + \sin t)y}{5} \Rightarrow \int \frac{dy}{y} = \int \frac{1}{5}(.5 + \sin t)\,dt$ $y(0) = 1$

$\ln y = \frac{1}{5} \cdot \frac{1}{2} t - \frac{1}{5}\cos t + C = \frac{1}{10}t - \frac{1}{5}\cos t + C$

$y = Y_0 e^{\frac{1}{10}t - \frac{1}{5}\cos t} \Rightarrow 1 = Y_0 e^{0 - \frac{1}{5}} \Rightarrow Y_0 = e^{1/5} \Rightarrow y(t) = e^{\frac{1}{50}t - \frac{1}{25}\cos t}$

$2 = e^{\frac{1}{50}t - \frac{1}{25}\cos t} \Rightarrow \ln 2 = \frac{1}{50}t - \frac{1}{25}\cos t$ $t = 33.56$ pop doubles

Suppose we start w/ $y(0) = 10$

$10 = Y_0 e^{-1/5} \Rightarrow Y_0 = 10e^{1/5} \Rightarrow y(t) = 10e^{\frac{1}{50}t - \frac{1}{25}\cos t}$ (no, it does not)

4. $T(0) = 200$ $T(1) = 190°$ $T_s = 70°$

$\frac{dT}{dt} = k(T - T_s) \Rightarrow \frac{dT}{T - 70} = k\,dt \Rightarrow \ln|T - 70| = kt + C \Rightarrow$

$T - 70 = T_0 e^{kt} \Rightarrow T(t) = T_0 e^{kt} + 70 \Rightarrow T(0) = T_0 + 700 = 200 \Rightarrow T_0 = 130$

$T(t) = 130e^{kt} + 70 \Rightarrow \frac{190 - 70}{130} = e^{k(1)} \Rightarrow k = -.08 \Rightarrow T(t) = 130e^{-.08t} + 70$

$150 = 130e^{-.08t} + 70 \Rightarrow -.08t = -.4855 \Rightarrow$ 6.07 minutes

5. Since $|v|$ is speed, call $|v| = s$ $\quad s(0)=0$, $g=32$

$mg - ks = ms' \Rightarrow g - \frac{k}{m}s = s'$ convert pounds to mass $\frac{180}{32} = 5.625$

$s' = 32 - \frac{.75}{5.625}s \Rightarrow s' = 32 - \frac{2}{15}s \Rightarrow s' = -\frac{2}{15}(s-240) \Rightarrow \frac{ds}{s-240} = -\frac{2}{15}dt$

$\ln|s-240| = -\frac{2}{15}t + C \Rightarrow s-240 = s_0 e^{-2/15 t} \Rightarrow s(t) = s_0 e^{-\frac{2}{15}t} + 240$

$s(0) = 0 \Rightarrow -240 = s_0 \Rightarrow s(t) = 240 - 240e^{-\frac{2}{15}t}$ Continues until $t=10$ when chute opens. $s(10) = 176.74$ acts like $s(0)$ for part 2.

$s' = 32 - \frac{12}{5.625}s \Rightarrow s' = 32 - \frac{32}{15}s = -\frac{32}{15}(s-15) \Rightarrow \int \frac{ds}{s-15} = \int -\frac{32}{15}dt$

$\ln|s-15| = -\frac{32}{15}t + C \Rightarrow s = s_0 e^{-\frac{32}{15}t} + 15$ using new $s(10)$

or $s(t) = s_0 e^{-\frac{32}{15}(t-10)} + 15$

$176.74 = s_0 e^0 + 15 \Rightarrow s_0 = 161.74$ $\quad s(t) = 161.74e^{-\frac{32}{15}(t-10)} + 15$

$$s(t) \text{ together} = \begin{cases} 240 - 240e^{-\frac{2}{15}t} & 0 \le t \le 10 \\ 161.74e^{-\frac{32}{15}(t-10)} & 10 < t \end{cases}$$ until he hits the ground.

limiting velocity w/ chute is 15 ft/sec.

to find distance fallen before parachute opens integrate speed.

$\int_0^{10} 240 - 240e^{-\frac{2}{15}t}dt = 240t + 1800e^{-\frac{2}{15}t}\Big|_0^{10} = 2400 + 1800e^{-\frac{20}{15}} - 1800$

≈ 1074.47 feet.

6. a. $y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$ $\quad (0,3) \cup (3,\infty)$ $\quad y(1)=2$

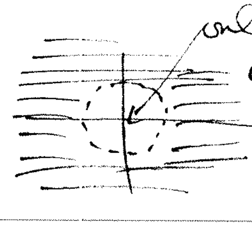
$\boxed{(0,3)}$

b. $y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$ $\quad (-\infty,-2) \cup (-2,2) \cup (2,\infty)$ $\quad y(-3)=1$

$\boxed{(-\infty,-2)}$

c. $y' = \sqrt{1-t^2-y^2}$. $1-t^2-y^2 \geq 0 \Rightarrow t^2+y^2 \le 1$ inside unit circle

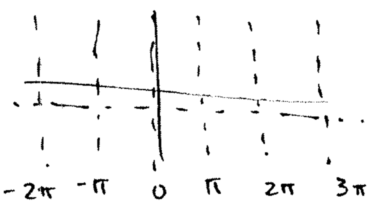
$\frac{\partial f}{\partial y} = \frac{-2y}{2\sqrt{1-t^2-y^2}}$ eliminates boundary



only ok inside circle

6d. $\frac{dy}{dt} = \frac{(\cot t)y}{1+y}$ $\quad$ t not defined for multiples of π
$y \neq -1$

$\frac{\partial f}{\partial y} = (\cot t)\left[\frac{1(1+y) - y(1)}{(1+y)^2}\right]$ same



ok everywhere else

7a. $M = 3x^2 - 2xy + 2$ $\quad$ $N = 6y^2 - x^2 + 3$

$\frac{\partial M}{\partial y} = -2x$ $\quad$ $\frac{\partial N}{\partial x} = -2x$ They match so is exact

$\int 3x^2 - 2xy + 2\,dx = x^3 - x^2y + 2x + f(y)$

$\int 6y^2 - x^2 + 3\,dy = 2y^3 - x^2y + 3y + g(x)$

$\psi(x,y): x^3 - x^2y + 2y^3 + 2x + 3y = C$

b. $M = 9x^2 + y - 1$ $\quad$ $N = -(4y - x) = -4y + x$

$\frac{\partial M}{\partial y} = 1$ $\quad$ $\frac{\partial N}{\partial x} = 1$ is exact

$\int 9x^2 + y - 1\,dx = 3x^3 + xy - \cancel{y} + f(y)$

$\int -4y + x\,dy = -2y^2 + xy + g(x)$

$\psi(x,y): 3x^3 - 2y^2 + xy - x = C$

c. $M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$

$\frac{\partial M}{\partial y} = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$

$N = xe^{xy}\cos 2x - 3$

$\frac{\partial N}{\partial x} = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$ is exact

$\int \underbrace{ye^{xy}\cos 2x - 2e^{xy}\sin 2x}_{\text{product rule for } e^{xy}\cos 2x} + 2x\,dx = e^{xy}\cos 2x + x^2 + f(y)$

$\int xe^{xy}\cos 2x - 3\,dy = e^{xy}\cos 2x - 3y + g(x)$

$\psi(x,y): e^{xy}\cos 2x + x^2 - 3y = C$

8a. $x^2y^3 + x(1+y^2)y' = 0 \qquad \mu(x,y) = \frac{1}{xy^3}$

$\frac{x^2y^3}{xy^3} + \frac{x(1+y^2)}{xy^3}y' = 0 \Rightarrow x\,dx + \frac{1+y^2}{y^3}dy = 0 \Rightarrow x\,dx + (y^{-3} + \frac{1}{y})dy = 0$

$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0$

$\int x\,dx = \frac{1}{2}x^2 + f(y) \qquad \int y^{-3} + \frac{1}{y}\,dy = \frac{-1}{2y^2} + \ln y + g(x)$

$\psi(x,y)$: $\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln y = C$

b. $y\,dx + (2x - ye^y)dy = 0 \quad \mu(x,y) = y \Rightarrow y^2dx + (2xy - y^2e^y)dy = 0$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y \qquad \int y^2dx = xy^2 + f(y) \quad \int 2xy - y^2e^y\,dy =$

$xy^2 - y^2e^y + 2ye^y - 2e^y + g(x)$

+/-	u	dv
+	y^2	e^y
-	$2y$	e^y
+	2	e^y
-	0	e^y

$\psi(x,y)$: $xy^2 - y^2e^y + 2ye^y - 2e^y = C$

c. $dx + (\frac{x}{y} - \sin y)dy = 0 \qquad \frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = \frac{1}{y} \qquad \ln\mu = \ln y \Rightarrow \mu = y$

$y\,dx + (x - y\sin y)dy = 0 \qquad \frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$ ✓

$\int y\,dx = xy + f(y) \qquad \int x - y\sin y\,dy = xy + y\cos y - \sin y + g(x)$

+/-	u	dv
+	y	$\sin y$
-	1	$-\cos y$
+	0	$-\sin y$

$\psi(x,y)$: $xy + y\cos y - \sin y = C$

d. $e^x dx + (e^x\cot y + 2y\csc y)dy = 0 \qquad \frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = e^x\cot y \qquad \ln\mu = \ln\sin y$

$\mu = \sin y$

$\ln(\mu(y)) = \int \frac{e^x\cot y}{e^x}dy = \int \cot y\,dy$

$e^x\sin y\,dx + (e^x\cos y + 2y)dy = 0$

$\frac{\partial M}{\partial y} = e^x\cos y \qquad \frac{\partial N}{\partial x} = e^x\cos y$

$\int e^x\sin y\,dx = e^x\sin y + f(y) \qquad \int e^x\cos y + 2y\,dy = e^x\sin y + y^2 + g(x)$

$\psi(x,y)$: $e^x\sin y + y^2 = C$