Math 2415, Exam #2, Summer 2014 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Solve the second order ordinary differential equations with constant coefficients. (10 points each)

   a. $y'' + y' - 2y = 0$

   $r^2 + r - 2 = 0$

   $(r+2)(r-1) = 0$

   $r = -2, r = 1$

   $y = c_1 e^{-2t} + c_2 e^{t}$

   b. $4y'' + 4y' + y = 0$

   $4r^2 + 4r + 1 = 0$

   $(2r+1)^2 = 0$

   $r = -1/2$ repeated

   $y = c_1 e^{-1/2 t} + c_2 t e^{-1/2 t}$

   c. $y'' + 4y' + 5y = 0$

   $r^2 + 4r + 5 = 0$

   $r = \frac{-4 \pm \sqrt{16 - 20}}{2} =$

   $= \frac{-4 \pm 2i}{2} = -2 \pm i$

   $y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

2. Find the value of the Wronskian using Abel's Theorem of the differential equation $(t-1)y'' - 3ty' + 4y = \sin t$. (7 points)

$$y'' - \underbrace{\frac{3t}{t-1}}_{p(t)} y' + \frac{4}{t-1} y = \frac{\sin t}{t-1}$$

$$t-1 \overline{)\,3t\,}\quad 3, \quad -3t+3, \quad 3$$

$$W = ce^{+\int \frac{3t}{t-1}dt} = ce^{\int 3 + \frac{3}{t-1}dt} = ce^{3t + 3\ln|t-1|} = ce^{3t}e^{\ln(t-1)^3} =$$

$$c(t-1)^3 e^{3t} \neq 0 \text{ for all } t \neq 1$$

3. Determine if the solutions $y_1 = t, y_2 = e^t, y_3 = te^t$ form a fundamental set by finding the value of the Wronskian. (8 points)

$$W = \begin{vmatrix} t & e^t & te^t \\ 1 & e^t & e^t + te^t \\ 0 & e^t & 2e^t + te^t \end{vmatrix} = t\left[(e^t)(2e^t + te^t) - e^t(e^t + te^t)\right] -$$

$$e^t\left[2e^t + te^t - 0\right] + te^t\left[e^t - 0\right] =$$

$$2te^{2t} + \cancel{t^2e^{2t}} - \cancel{te^{2t}} - \cancel{t^2e^{2t}} - 2e^{2t} - te^{2t} + \cancel{te^{2t}}$$

$$te^{2t} - 2e^{2t} = e^{2t}(t-2) \neq 0 \text{ for all } t \neq 2$$

4. Find the solution to the Cauchy-Euler equation for the ODE $t^2y'' + ty' + y = 0, y(1) = 1, y'(1) = 3$. (12 points)

$$t^n = y \qquad y' = nt^{n-1} \qquad y'' = n(n-1)t^{n-2}$$

$$t^2(n)(n-1)t^{n-2} + nt \cdot t^{n-1} + t^n = 0$$

$$t^n[n(n-1) + n + 1] = 0 \Rightarrow n^2 - \cancel{n} + \cancel{n} + 1 = 0 \Rightarrow n^2 = -1$$

$$n = \pm 1$$

$$y = c_1 \cos(\ln t) + c_2 \sin(\ln t) \Rightarrow y = \cos(\ln t) + 3\sin(\ln t)$$

$$1 = c_1$$

$$y' = \frac{-c_1 \sin(\ln t)}{t} + \frac{c_2 \cos(\ln t)}{t}$$

$$3 = c_2$$

5. Use the method of reduction of order to find the second solution to the differential equation $xy'' + (2+2x)y' + 2y = 0$ for the given solution $y_1 = \frac{1}{x}$. (14 points)

$y_2 = \frac{v}{x}$

$y_2' = \frac{v'}{x} + \frac{-v}{x^2} = \frac{v'}{x} - \frac{v}{x^2}$

$y_2'' = \frac{v''}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}$

$x\left(\frac{v''}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}\right) + (2+2x)\left(\frac{v'}{x} - \frac{v}{x^2}\right) + 2\left(\frac{v}{x}\right) = 0$

$v'' - \cancel{\frac{2v'}{x}} + \cancel{\frac{2v}{x^2}} + \cancel{\frac{2v'}{x}} - \cancel{\frac{2v}{x^2}} + 2v' - \cancel{\frac{2v}{x}} + \cancel{\frac{2v}{x}} = 0$

$v'' + 2v' = 0$

$v'' = -2v'$

$u' = -2u$

$u = v'$

$v'' = u'$

$\int \frac{u'}{u} = \int -2\, dx$

$\ln u = -2x$

$u = e^{-2x} = v'$

$v = \int e^{-2x} dx =$

$-\frac{1}{2}e^{-2x}$

$y_2 = \frac{e^{-2x}}{x}$

$y = \frac{c_1}{x} + \frac{c_2 e^{-2x}}{x}$

6. Find the particular solution to the nonhomogeneous differential equation $y'' + y' + 4y = 2\sinh t$ using the method of undetermined coefficients. (12 points)

$y_p(t) = A\sinh t + B\cosh t$

$y_p'(t) = A\cosh t + B\sinh t$

$y_p''(t) = A\sinh t + B\cosh t$

Sinht is not a solution to the original system.

$A\sinh t + B\cosh t + A\cosh t + B\sinh t + 4A\sinh t + 4B\cosh t = 2\sinh t$

$5A + B = 2 \Rightarrow 5(-5B) + B = 2 \Rightarrow -25B + B = -24B = 2 \Rightarrow B = -1/12$

$5B + A = 0 \Rightarrow A = -5B$

$A = 5/12$

$y_p(t) = \frac{5}{12}\sinh t - \frac{1}{12}\cosh t$

7. Suppose that the solutions to a second order differential equation are $y_1(t) = e^t, y_2(t) = e^{-2t}$. If the forcing term on the nonhomogeneous ODE is $F(t) = t^2e^{-t} + e^{-2t} + 4\sin t$, state your initial Ansatz for the method of undetermined coefficients (you do not need to solve for any of the coefficients, just state where you would start). (6 points)

$$Y(t) = (At^2 + Bt + C)e^{-t} + Dte^{-2t} + E\sin t + F\cos t$$

8. Give an example of three functions that would need to be solved by the method of variation of parameters and cannot be solved by the method of undetermined coefficients. (6 points)

$\tan x$

$\ln x$

$\sqrt{x}$ answers will vary

9. Describe when a 'beat' phenomenon occurs in a forced spring problem. (6 points)

beats occur when steady-state oscillations in a system (undamped or forcing) are close in frequency with the period of the beat related to the difference in frequency. The closer the frequencies are the larger the beat period

10. Suppose that a mass of 20 kg stretches a spring 5 cm. Suppose that the mass is attached to a viscous damper with a damping constant of 400 Ns/m. If the mass is pulled down an additional two centimeters and then released, find the differential equation and initial conditions to be used to solve for the position of the system. (You do not need to solve the equation, just set it up.) (10 points)

$m = 20$

$k = \frac{20 * 9.8}{.05} = 400 * 9.8 = 3920$ $\quad \gamma = 400$

$20y'' + 400y' + 3920y = 0$

$y(0) = -.02$

$y'(0) = 0$

11. For each of the proposed solutions to a spring problem, describe any salient characteristics of the system. For instance, what is the transient solution (if it exists)? What is the steady state solution (if it exists)? Is the system undamped, underdamped, critically damped or overdamped? What is the frequency (or quasi-frequency) of the system? Does the system experience resonance or beats? [Hint: it may help to graph the system.] (8 points each)

a. $y(t) = \underbrace{e^{-t}\sin t + 3e^{-t}\cos t}_{\text{transient}} + \underbrace{t\sin t}_{\text{steady state}}$

underdamped

quasi-freq. 1 freq = 1

no beats or resonance (per se)

b. $y(t) = \underbrace{e^{-3t} - te^{-3t}}_{\text{transient}} + \underbrace{\cos(2t)}_{\text{steady state}}$

Critically damped

no frequency frequency = 2

no resonance or beats

c. $y(t) = 11\sin\left(\frac{5}{6}t\right) - 7\cos\left(\frac{5}{6}t\right) + \cos\left(\frac{3}{4}t\right)$

no transient solution
undamped

2 frequencies $= \frac{5}{6}, \frac{3}{4}$

does experience beats but no resonance

12. Find the solution to the boundary value problem $y'' + y = 0, y'(0) = 1, y(L) = 0$. (8 points)

$r^2+1=0$

$r = \pm i$

$y = c_1 \cos t + c_2 \sin t$

$y' = -c_1 \sin t + c_2 \cos t$

$1 = c_2$

$0 = c_1 \cos L + \sin L$

$\frac{-\sin L}{\cos L} = c_1 \Rightarrow c_1 = -\tan L$

$y = (-\tan L)\cos t + \sin t$