Math 2366, Exam #2, Summer 2014 Name KEY

Instructions: Show all work. Justify answers as completely as possible. If you are asked to prove something, mere computation is not enough. You must explain your reasoning. Be sure to state your conclusion clearly. Incomplete work or justification will not receive full credit. Use exact answers unless specifically asked to round.

1. Use mathematical induction to prove that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. (12 points)

base case: $\sum_{i=1}^{1} i^3 = 1^3 = 1$ $\quad \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$ ✓

suppose true for $n \geq 1$ show true for $n+1$

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

$$= \frac{(n+1)^2(n^2 + 4(n+1))}{4} = \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

$$= \left[\frac{(n+1)(n+2)}{2}\right]^2$$ and according to formula

$$\sum_{i=1}^{n+1} i^3 = \left[\frac{(n+1)(n+1+1)}{2}\right]^2 = \left[\frac{(n+1)(n+2)}{2}\right]^2$$ which was proved.

QED.

2. Find $f(2), f(3), f(4), f(5)$ if $f(0) = 1, f(1) = 2$, and f is defined recursively as $f(n+1) = f(n)^2 - n$. (6 points)

$f(2) = 2^2 - 1 = 4 - 1 = 3$

$f(3) = 3^2 - 2 = 9 - 2 = 7$

$f(4) = 7^2 - 3 = 49 - 3 = 46$

$f(5) = 46^2 - 4 = 2112$

3. Find a recursive definition of the sequence of numbers that are congruent to 2 modulo 3. (5 points)

$2 \bmod 3 = \{2, 5, 8, 11, \dots\}$

$a_{n+1} = a_n + 3$ or $f(n+1) = f(n) + 3$

$a_0 = 2$ $\quad f(0) = 2$

4. Find the number of telephone numbers in the 614 area code, assuming that the first digit after the area code cannot be 0 or 1. (5 points)

8×10^6

5. How many strings of 12 alphanumeric characters are possible if upper and lowercase are different, and no letter can be repeated? (5 points)

$26 + 26 + 10 = 62$

$62P12 \approx 1.0347 \times 10^{21}$

6. How many positive integers not exceeding 1000 are divisible by either 3 or 7? (6 points)

$333 + 142 - 47 = 428$

7. Show that if there are 60 students in a class, there are at least three of them with the same first initial. (6 points)

Since there are 26 letters

$\frac{60}{26} = 2\frac{8}{26}$

So every letter has at least 2 students w/ the same letter in the "worst case" scenario, but there are still 8 students left, so at least 8 students must have the same first initial as 2 other students, thus 3 students have the same first initial.

8. Find the values of the following expressions by hand. (3 points each)
   a. P(13,4) $13 \cdot 12 \cdot 11 \cdot 10 = 17,160$

   b. C(21,15) $\frac{21!}{15!6!} = 54,264$

   c. $\binom{8}{5}$ $\frac{8!}{5!3!} = 56$

9. How many strings of the letters ABCDEFGHIJKLM have the string KGH if repetition is not allowed? (5 points)

$10!(1)(11) = 39,916,800$

10. Find the expansion of $(3x - 2y)^4$. (5 points)

$81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$

11. A bagel shop has onion, everything, egg, raisin, plain, poppy seed, salted, rye, chocolate chip, cinnamon swirl, asiago cheese and cranberry bagels. How many ways are there to choose a baker's dozen bagels? (6 points)

$$\binom{12+13-1}{13} = \binom{24}{13} = 2{,}496{,}144$$

12. How many strings can be made from the letters in the words YELLOW WEAVER? (5 points)

Y-1
E-3
L-2
O-1
W=2
A-1
V-1
R-1

$$\frac{12!}{3!\,2!\,2!\,(1!)^5} = 19{,}958{,}400$$

13. Find the next seven elements in lexicographic order for the permutation 1623547. (8 points)

1623574
1623745
1623754
1624357
1624375
1624537
1624573

14. What is the probability that a five-card poker hand contains a full house (i.e. two of a kind and a three of kind)? (6 points)

$$\frac{(13)\binom{4}{2}\,12\binom{4}{3}}{\binom{52}{5}} \approx .00144$$

15. What is the probability that a random alphanumeric password (where case does NOT matter) contains only even numbers and vowels? (7 points)

$26+10=36$

$5+5=10$

$\frac{10}{36} \approx .278$

$\left(\frac{5}{18}\right)^n$

where n is the # of characters

16. What is the probability that a fair coin is flipped 15 times and results in five heads? (5 points)

$\binom{15}{5}\left(\frac{1}{2}\right)^{15} = .0916$

17. What is the probability that a fair die is rolled 10 times and there are fewer than 3 twos? (6 points)

$\binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} + \binom{10}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9 + \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$

$\approx .775$

18. Suppose that you flip an unfair coin (with a probability of heads being 2/3). What is the expected number of tails in 20 flips? (5 points)

$\frac{1}{3}(20) = \frac{20}{3} \approx 6.7$

19. Based on the table below, determine if the two variables are independent. If they are, explain how you know. If they are not, explain why not. (18 points)

$p(x,y)$	0	1	2	
0	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{10}$	$17/40$
1	$\frac{3}{16}$	$\frac{1}{25}$	$\frac{1}{16}$	$29/100$
2	$\frac{7}{50}$	$\frac{3}{25}$	$\frac{1}{40}$	$57/200$
	$161/400$	$57/200$	$3/16$	

$E(X) = \frac{57}{200} + 2(3/16) = .66$ $\quad E(Y) = 29/100 + 2(57/200) = .86$

$E(XY) = 1/25 + 2(1/16) + 2(3/25) + 4(1/40) = .505$

$E(X)E(Y) = .5676$ not independent since $E(XY) \neq E(X)E(Y)$

20. Explain Chebyshev's inequality in your own words. (4 points)

if X is a random variable on a space S, and r is real, then the probability that the distance from the expected value of some result of the random variables is greater than r is approximately (less than) $V(X)/r^2$.

21. If $E(X) = 4.5$ and $E(Y) = 9.7$, and X and Y are independent, find the values of the following expressions. (3 points each)

a. $E(X+Y)$ $= E(X) + E(Y) = 4.5 + 9.7 = 14.2$

b. $E(11X+30)$ $= 11E(X) + 30 = 11(4.5) + 30 = 79.5$

c. $E(XY)$ $= E(X)E(Y) = 4.5 * 9.7 = 43.65$