Name KEY

Math 2568, Final Exam -- Part 1, Summer 2013

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Find the polynomial of the form $a_2t^2 + a_1t + a_0 = p(t)$ that passes through the points (1,2), (2,8), (3, 16). Write the system of equations and matrix for the system. Give the resulting polynomial. You will need to row reduce the system to echelon form by hand. (20 points)

$$a_2 + a_1 + a_0 = 2$$
$$4a_2 + 2a_1 + a_0 = 8$$
$$9a_2 + 3a_1 + a_0 = 16$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 8 \\ 9 & 3 & 1 & 16 \end{array}\right] \quad \begin{array}{l} -4R_1+R_2 \to R_2 \\ -9R_1+R_3 \Rightarrow R_3 \end{array}$$

$$\begin{array}{cccc} -4 & -4 & -4 & -8 \\ -9 & -9 & -9 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & 0 \\ 0 & -6 & -8 & -2 \end{array}\right] \quad \begin{array}{l} -\frac{1}{2}R_2 \to R_2 \\ -3R_2+R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\begin{array}{cccc} 0 & 6 & 9 & 0 \end{array}$$

$a_0 = -2$

$a_1 + 3/2\, a_0 = 0 \Rightarrow a_1 + 3/2(-2) = 0 \Rightarrow a_1 - 3 = 0 \Rightarrow a_1 = 3$

$a_2 + a_1 + a_0 = 2 \Rightarrow a_2 + 3 + (-2) = 2 \Rightarrow a_2 + 1 = 2 \Rightarrow a_2 = 1$

$\boxed{p(t) = t^2 + 3t - 2}$

2. Find the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 4 & 0 \end{bmatrix}$ by any means. (12 points)

$$+(-3)\begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} + 0 = -3(12+5) - 2(4-10)$$

$$-3(17) - 2(-6) = -51 + 12 = \boxed{-39}$$

3. Find the QR factorization of the matrix A, given that $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$ and $Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$. In other words, find R. (8 points)

$Q^T A = R$ $\quad \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix} =$

$\begin{bmatrix} +4/7 + 25/7 + 4/7 + 16/7 & -6/7 + 35/7 + -4/7 + 24/7 \\ -10/7 + 10/7 - 8/7 + 8/7 & 15/7 + 14/7 + 8/7 + 12/7 \end{bmatrix} =$

$\begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix} = R$

4. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}$. Be sure to clearly indicate the characteristic equation, and which eigenvalues and eigenvectors go together. (20 points)

$\begin{bmatrix} 1-\lambda & -3 \\ -4 & 5-\lambda \end{bmatrix} \Rightarrow (1-\lambda)(5-\lambda) - 12 = 5 - 6\lambda + \lambda^2 - 12 =$

$\lambda^2 - 6\lambda - 7 = 0 \quad (\lambda - 7)(\lambda + 1) = 0 \quad \lambda = 7, \lambda = -1$

$\lambda_1 = 7 \begin{bmatrix} 1-7 & -3 \\ -4 & 5-7 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ -4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad 2x_1 + x_2 = 0$, $x_1 = -\frac{1}{2}x_2$ $\quad \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_2 = -1 \begin{bmatrix} 1-(-1) & -3 \\ -4 & 5-(-1) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \Rightarrow 2x_1 = 3x_2$, $x_1 = 3/2 x_2$ $\quad \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

5. Consider the orthogonal basis for $\mathbb{R}^3$ given by $\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix},\begin{bmatrix}0\\3\\1\end{bmatrix},\begin{bmatrix}-10\\-1\\3\end{bmatrix}\right\}$. Use the property of Orthogonality to find the coordinate representation of the vector $\vec{x}=\begin{bmatrix}6\\2\\4\end{bmatrix}$ in this basis. [Hint: no matrices are required.] (15 points)

$$c_1 = \frac{6+(-2)+12}{1+1+9} = \frac{16}{11}$$

$$c_2 = \frac{0+6+4}{0+9+1} = \frac{10}{10} = 1$$

$$c_3 = \frac{-60-2+12}{100+1+9} = \frac{-50}{110} = -\frac{5}{11}$$

$$[\vec{x}]_B = \begin{bmatrix}16/11\\1\\-5/11\end{bmatrix}$$

6. Compute 2A - B^T given $A=\begin{bmatrix}2&1&5\\-1&0&6\\2&1&-1\end{bmatrix}$ and $B=\begin{bmatrix}-1&7&1\\0&2&8\\-6&1&0\end{bmatrix}$ (8 points)

$$2A=\begin{bmatrix}4&2&10\\-2&0&12\\4&2&-2\end{bmatrix} \quad B^T=\begin{bmatrix}-1&0&-6\\7&2&1\\1&8&0\end{bmatrix}$$

$$2A-B^T = \begin{bmatrix}4-(-1)&2-0&10-(-6)\\-2-7&0-2&12-1\\4-1&2-8&-2-0\end{bmatrix} = \begin{bmatrix}5&2&16\\-9&-2&11\\3&-6&-2\end{bmatrix}$$

7. Find the inverse of $C = \begin{bmatrix} 1 & 5 \\ 3 & -2 \end{bmatrix}$. (8 points)

$$C^{-1} = \frac{1}{-2-15}\begin{bmatrix} -2 & -5 \\ -3 & 1 \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -2 & -5 \\ -3 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2/17 & 5/17 \\ 3/17 & -1/17 \end{bmatrix}$$

8. Show that the polynomials $f(t) = 1 - 2t$, and $g(t) = 8 + 3t$ are orthogonal under the inner product $< f, g > = \int_{-2}^{2} f(t)g(t)dt$. (10 points)

$$\int_{-2}^{2} (1-2t)(8+3t)dt = \int_{-2}^{2} 8+3t-16t-6t^2\, dt$$

$\int_{-a}^{a} \text{odd} = 0 \Rightarrow \int_{-2}^{2} 8-6t^2\,dt + \underbrace{\int_{-2}^{2} 3t-16t\,dt}_{\to 0} = \int_{-2}^{2} 8-6t^2\,dt$

$\int_{-a}^{a} \text{even} = 2\int_{0}^{a} \text{even} \Rightarrow 2\int_{0}^{2} 8-6t^2\,dt = 2\left[8t-2t^3\right]_0^2 =$

$2[16-16] = 0$

Yes, they are orthogonal.

9. Determine if the following sets of vectors are linearly independent by inspection. Justify your answer in each case. (5 points each)

a. $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ Zero vector, not independent

b. $\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\\3\end{bmatrix}, \begin{bmatrix}4\\0\\0\\5\end{bmatrix}, \begin{bmatrix}3\\3\\2\\1\end{bmatrix}, \begin{bmatrix}6\\7\\-1\\1\end{bmatrix}\right\}$ five vectors in R^4, not independent

c. $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ 2 vectors, not multiples, independent

10. Determine if each statement is True or False. (3 points each)

a. T F If vectors $\vec{v_1}, \vec{v_2}, \dots \vec{v_p}$ span a subspace W and if $\vec{x}$ is orthogonal to each $\vec{v_j}$ for j=1…p, then $\vec{x}$ in $W^{\perp}$.

b. T F If $\vec{y}$ is in a subspace W, then the orthogonal projection of $\vec{y}$ onto W is $\vec{y}$ itself.

c. T F A trivial solution means that a non-zero solution exists.

d. T F Both $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$ and $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are matrices in echelon form.

no ok

e. T F If A is a 3x4 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ can be one-to-one and onto.

f. T F If a system of equations has a free variable then it has a unique solution.

g. T F If A is a $n \times n$ matrix, then A is invertible.

h. T F If two vectors are orthogonal, they are linearly independent.

i. T F The set of all odd functions is an example of a vector space. no zero vector

j. T F The vector space P_n and R^{n+1} are isomorphic.

k. T F The null space of a matrix is a subspace of the codomain of the matrix.

= range

l. T F The second standard basis vector $\vec{e_2}$ in $\mathbb{R}^4$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

m. T F If an eigenvalue is repeated p times in the characteristic equations of a matrix, then there will always be p linearly independent eigenvectors corresponding to that eigenvalue.

n. T F A matrix is not invertible if and only if 0 is an eigenvalue of A.

o. T F If the columns of A are linearly independent, then the equation $A\vec{x} = \vec{b}$ has exactly one least-squares solution.

p. T F A least-squares solution of $A\vec{x} = \vec{b}$ is the point in the column space of A closest to $\vec{b}$.

Name____

Math 2568, Final Exam -- Part 2, Summer 2013

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Find a least squares solution for the set of points $\{(1,0.7),(2.1,2.5),(2.2,3),(3.2,4.8),(4.9,6.4),(5.6,9.7)\}$ to satisfy the equation $y=\beta_0+\beta_1 x+\beta_2 x^2$. Be sure to write the matrices employed, any equations, and the final regression function for y. (15 points)

$$A=\begin{bmatrix}1&1&1\\1&2.1&2.1^2\\1&2.2&2.2^2\\1&3.2&3.2^2\\1&4.9&4.9^2\\1&5.6&5.6^2\end{bmatrix}\quad \vec{b}=\begin{bmatrix}.7\\2.5\\3\\4.8\\6.4\\9.7\end{bmatrix}\qquad (A^TA)^{-1}A^T\vec{b}=\vec{X}_{11}$$

$$A=\begin{bmatrix}1&1&1\\1&2.1&4.41\\1&2.2&4.84\\1&3.2&10.24\\1&4.9&24.01\\1&5.6&31.36\end{bmatrix}\qquad \vec{X}_{11}=\begin{bmatrix}-.433...\\1.269...\\.0735...\end{bmatrix}=\begin{bmatrix}\beta_0\\\beta_1\\\beta_2\end{bmatrix}$$

$$\boxed{y\approx .0735x^2+1.27x-.433}$$

2. Given the vectors $\vec{b_1}=\begin{bmatrix}4\\1\\2\\5\end{bmatrix}$ *and* $\vec{b_2}=\begin{bmatrix}1\\-4\\-5\\2\end{bmatrix}$, find two more vectors orthogonal to these (and each other) to make an orthogonal basis for $\mathbb{R}^4$. (15 points)

$$A=\begin{bmatrix}4&1&2&5\\1&-4&-5&2\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&3/17&22/17\\0&1&22/17&-3/17\end{bmatrix}\qquad \begin{aligned}x_1&=-3/17x_3-22/17x_4\\x_2&=-22/17x_3+3/17x_4\end{aligned}$$

let $x_4=0$, $x_3=17$ $\quad\vec{b_3}=\begin{bmatrix}-3\\-22\\17\\0\end{bmatrix}$

$$B=\begin{bmatrix}4&1&2&5\\1&-4&-5&2\\-3&-22&17&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0&22/17\\0&1&0&-3/17\\0&0&1&0\end{bmatrix}\qquad \begin{aligned}x_1&=-22/17x_4\\x_2&=3/17x_4\\x_3&=0\end{aligned}\qquad \text{let } x_4=17$$

$$\vec{b_4}=\begin{bmatrix}-22\\3\\0\\17\end{bmatrix}\qquad \text{basis }\left\{\begin{bmatrix}4\\1\\2\\5\end{bmatrix},\begin{bmatrix}1\\-4\\-5\\2\end{bmatrix},\begin{bmatrix}-3\\-22\\17\\0\end{bmatrix},\begin{bmatrix}-22\\3\\0\\17\end{bmatrix}\right\}$$

3. The set $H = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$. Use the Gram-Schmidt Process to make an orthogonal basis, and then normalize it. (15 points)

$$\vec{b_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \vec{b_2} = \vec{v_2} - \text{proj}_{v_1} \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0+0-1}{1+0+1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$b_3 = \vec{v_3} - \text{proj}_{v_1} \vec{v_3} - \text{proj}_{b_2} \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \frac{1+0+2}{1+0+1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1+2+2}{1+4+1} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3 \\ -3/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 - 3/2 - 3/2 \\ 1 - 0 - 3 \\ -2 + 3/2 + 3/2 \end{bmatrix} = \begin{bmatrix} 1-3 \\ 1-3 \\ -2+3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ +1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

4. Given the basis of $W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$, and the vector $\vec{y} = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 0 \end{bmatrix}$ decompose this vector into $\vec{y} = \vec{y_\parallel} + \vec{y_\perp}$ with $\vec{y_\parallel} = proj_W \vec{y}$. (15 points)

$$\vec{Y_{\parallel}} = \frac{3+4+0+0}{1+1} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{0+0+-2+0}{0+0+1+1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \frac{7}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-2}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} 7/2 - 0 \\ 7/2 - 0 \\ 0 - 1 \\ 0 - (-1) \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \\ -1 \\ 1 \end{bmatrix} = \vec{Y_{\parallel}}$$

$$\vec{Y_\perp} = \vec{Y} - \vec{Y_\parallel} = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 7/2 \\ 7/2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1 \\ -1 \end{bmatrix}$$

5. **Use an inverse matrix** to solve $\begin{cases} x_1 \quad\quad - 2x_3 = 9 \\ -5x_1 + x_2 + 3x_3 = -8 \\ 2x_1 - x_2 + 6x_3 = 1 \end{cases}$. Give the inverse matrix used. You should write the matrix equation to be solved, the solution with the inverse matrix in equation form, and the final solution in vector form. (8 points)

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -5 & 1 & 3 \\ 2 & -1 & 6 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 3 & 2/3 & 2/3 \\ 12 & 10/3 & 7/3 \\ 1 & 1/3 & 1/3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 9 \\ -8 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$A^{-1}\vec{b} = \vec{x} = \begin{bmatrix} 67/3 \\ 251/3 \\ 20/3 \end{bmatrix}$$

6. The following are short answer questions. Always provide justification for any answers. You may use examples as part of your explanations, but if you are asked to "explain" your answer must contain **words.** (6 points each)

   a. Give an example of a 6x4 matrix with a non-trivial solution.

any 6x4 matrix w/ a free variable

e.g. $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable

   b. Explain how multiplying a matrix by a constant changes the determinant.

multiplying a matrix by a constant k changes the determinant by k^n since multiplying a matrix row by a constant changes the determinant by k and there are n rows.

c. Explain why the vectors in the null space of an mxn matrix must be in $\mathbb{R}^n$.

vectors in the nullspace are vectors which, when multiplied by a matrix must produce $\vec{0}$ as its output. in order to do the multiplication w/ an $m \times n$ matrix, the vectors in null A must be $n \times 1$, thus in $\mathbb{R}^n$.

d. Explain why if an nxn matrix has n pivot positions, it cannot have 0 as an eigenvalue.

If a matrix has n pivot positions then it has a non-zero determinant, since it is row equivalent to the identity. the identity is a diagonal matrix and so its eigenvalues are on its diagonal. While performing row operations can change the scaling of eigenvalues it cannot change non-zero eigenvalues to zero or zero eigenvalues to non-zero. any matrix w/ zero as an eigenvalue has a nullspace and so will not have n pivots.

e. Explain why the properties of linear transformations and vector spaces are so similar.

a linear transformation maps from one vector space into another, and so the properties of the vector space must be preserved in the transformation.

f. Define the term *isomorphism*. Give an example of two spaces that are isomorphic to one another.

an isomorphism is a mapping which is both one-to-one and onto that maps one space into another of identical dimension. $\mathbb{P}_3$ and $\mathbb{R}^4$ are isomorphic and both are isomorphic to the set of 2x2 matrices.

g. Define the term *dimension*. What does it mean for a space to have 3 dimensions? What does it mean for a space to have infinite dimensions?

The term dimension refers to the number of basis vectors that define a space. $\mathbb{R}^3$ is a space w/ 3 dimensions because any basis of $\mathbb{R}^3$ must have 3 vectors in it. A space is infinitely dimensioned when there is no finite set of basis vectors that can span it. The set of all polynomials, for instance, because no matter the degree you choose, you can always find a higher degree polynomial not yet in the space.

h. What is the *eigenspace* of a matrix?

the eigenspace of a matrix is the space spanned by all the eigenvectors of a matrix.

i. Explain the relationship between a vector $\vec{y}$ in $\mathbb{R}^n$, W a subspace of $\mathbb{R}^n$, $\vec{v}, \vec{y_{\parallel}}$ which are vectors in W, as described by the Best Approximation Theorem.

$\vec{y}, \vec{v}, \vec{y_{\parallel}}$ are all vectors in $\mathbb{R}^n$. $\vec{v}, \vec{y_{\parallel}}$ in W.

according to the best approximation theorem $\vec{y_{\parallel}}$ is the closest vector to $\vec{y}$ in W such that

$\|\vec{y} - \vec{y_{\parallel}}\| < \|\vec{y} - \vec{v}\|$ for any other $\vec{v}$ in W where $\vec{v} \neq \vec{y_{\parallel}}$.