Math 2568, Exam #1 – Part 1, Summer 2013 Name KEY

Instructions: You may not use a calculator for this part of the exam. You must show work or provide justification to receive credit (partial or otherwise) for answers. Use exact answers.

1. Given the system of equations $\begin{cases} x_1 - 2x_2 + 4x_3 + 5x_4 = 10 \\ -x_1 + x_2 - 3x_3 + x_4 = -13 \end{cases}$, write the *system* as:

   a. An augmented matrix (3 points)

$$\left[\begin{array}{cccc|c} 1 & -2 & 4 & 5 & 10 \\ -1 & 1 & -3 & 1 & -13 \end{array}\right]$$

   b. A vector equation (3 points)

$$x_1\begin{bmatrix}1\\-1\end{bmatrix} + x_2\begin{bmatrix}-2\\1\end{bmatrix} + x_3\begin{bmatrix}4\\-3\end{bmatrix} + x_4\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}10\\-3\end{bmatrix}$$

   c. A matrix equation. (3 points)

$$\begin{bmatrix}1 & -2 & 4 & 5\\-1 & 1 & -3 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}10\\-13\end{bmatrix}$$

   d. Solve the system using the augmented matrix and row operations. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form (recall that for parametric form, you need reduced echelon form). (8 points)

$R_1 + R_2 \to R_2$ $\left[\begin{array}{cccc|c} 1 & -2 & 4 & 5 & 10 \\ 0 & -1 & 1 & 6 & -3 \end{array}\right]$ $-R_2 \to R_2$

$\left[\begin{array}{cccc|c} 1 & -2 & 4 & 5 & 10 \\ 0 & 1 & -1 & -6 & 3 \end{array}\right]$ $2R_2 + R_1 \to R_1$ $\left[\begin{array}{cccc|c} 1 & 0 & 2 & -7 & 16 \\ 0 & 1 & -1 & -6 & 3 \end{array}\right]$

$0 \quad 2 \quad -2 \quad -12 \quad 6$

$x_1 = -2x_3 + 7x_4 + 16$

$x_2 = x_3 + 6x_4 + 3$

$x_3 = \text{free} = x_3$

$x_4 = \text{free} = x_4$

$$\vec{x} = x_3\begin{bmatrix}-2\\1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}7\\6\\0\\1\end{bmatrix} + \begin{bmatrix}16\\3\\0\\0\end{bmatrix}$$

2. Given $A = \begin{bmatrix} 8 & -4 \\ 9 & -5 \end{bmatrix}$, find A^{-1}, either by employing the formula, or by the algorithm. (8 points)

$$\frac{1}{-40+36}\begin{bmatrix} -5 & 4 \\ -9 & 8 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -5 & 4 \\ -9 & 8 \end{bmatrix} = \begin{bmatrix} 5/4 & -1 \\ 9/4 & -2 \end{bmatrix} = A^{-1}$$

Check $\begin{bmatrix} 8 & -4 \\ 9 & -5 \end{bmatrix}\begin{bmatrix} 5/4 & -1 \\ 9/4 & -2 \end{bmatrix} = \begin{bmatrix} 10-9 & -8+8 \\ 45/4-45/4 & -9+10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

3. Given $A = \begin{bmatrix} 2 & -1 \\ -6 & 0 \\ 1 & 3 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} 0 & 5 & -1 \\ 4 & -2 & 0 \end{bmatrix}_{2\times 3}$, $C = \begin{bmatrix} 9 \\ -8 \\ 4 \end{bmatrix}_{3\times 1}$, $D = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 2 \\ -1 & 5 & 4 \end{bmatrix}_{3\times 3}$ compute the following, if possible. If the combination is not possible, briefly explain why. (5 points each)

a) AB

$$\begin{bmatrix} -4 & 12 & -2 \\ 0 & -30 & 6 \\ 12 & -1 & -1 \end{bmatrix}$$

b) BA

$$\begin{bmatrix} -31 & -3 \\ 20 & -4 \end{bmatrix}$$

c) C^TC

$= 81+64+16 = [161]$

1×1

matrix

aka, real #

f) $3D - \lambda I_3$

$$\begin{bmatrix} 3 & -9 & -3 \\ 0 & 0 & 6 \\ -3 & 15 & 12 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -9 & -3 \\ 0 & -\lambda & 6 \\ -3 & 15 & 12-\lambda \end{bmatrix}$$

4. Determine if the following sets of vectors are linearly independent. Justify your answer in each case. (5 points each)

a. $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ dependent $\vec{0}$ vector included

b. $\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\2\\0\\3\end{bmatrix},\begin{bmatrix}4\\0\\0\\5\end{bmatrix},\begin{bmatrix}3\\3\\2\\1\end{bmatrix},\begin{bmatrix}6\\7\\-1\\1\end{bmatrix}\right\}$ dependent. 5 vectors always dependent in $\mathbb{R}^4$

c. $\left\{\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ independent. not multiples of each other sufficient for 2 vectors

5. Consider the linear transformation defined by $T: \vec{x} \mapsto A\vec{x}$, with $A = \begin{bmatrix}1 & 2 & 0 & -2\\0 & -3 & 0 & 3\\9 & 4 & 2 & 4\\3 & -2 & 2 & 1\\1 & -1 & 5 & 0\end{bmatrix}$. (8 points)

a. What is the domain of T? Give an example of a vector in the domain.

$\mathbb{R}^4$ any vector of the form $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ such as $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$

b. What is the range (codomain) of T? Give an example of a vector in the range?

$\mathbb{R}^5$ (a subspace of it at least)

$A\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}-3\\6\\39\\9\\14\end{bmatrix}$

note: choose a vector in domain & transform it to avoid an inconsistent solution
not all vectors in $\mathbb{R}^5$ are obtainable

6. Determine if each statement is True or False. (2 points each)

a. T F Linear transformations can be represented as matrices in all cases.

think of derivatives $\neq$ matrix

b. T F A trivial solution means that a non-zero solution exists.

trivial means zero only

c. T F If A is a 4x3 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ can be one-to-one and onto.

can be one or the other but not both unless matrix is square.

d. T F Matrix multiplication is commutative

$AB \neq BA$ generally

e. T F If a matrix is nxn, and there exists a C matrix such that CA=I, then A and C are commutative. true since $C = A^{-1}$ and $AA^{-1} = I$ also

f. T F If a transformation is onto, then there exists a unique $\vec{x}$ in the domain that maps onto every $\vec{b}$ in the range. that is one-to-one

onto: every $\vec{b}$ has at least one $\vec{x}$ that maps onto it

g. T F If AB=AC, then B may not be equal to C.

h. T F Both $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$ and $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are matrices in echelon form.

not echelon form

i. T F If A and B are row equivalent, then they have the same reduced echelon form.

j. T F A set of two vectors in R^3 must be linearly independent.

Could be multiples

k. T F A homogeneous equation is always consistent.

l. T F Row reducing steps are reversible. allowable ones, yes.

m. T F A 5x6 matrix has six rows. 6 columns

n. T F If every column of an augmented matrix has a pivot, the system is consistent. the last column's pivot makes it inconsistent

o. T F Transposing a matrix does not change its size.

not if square, but if A is 4x3, then

A^T is 3x4.

Math 2568, Exam #1 – Part 2, Summer 2013 Name ______________________________

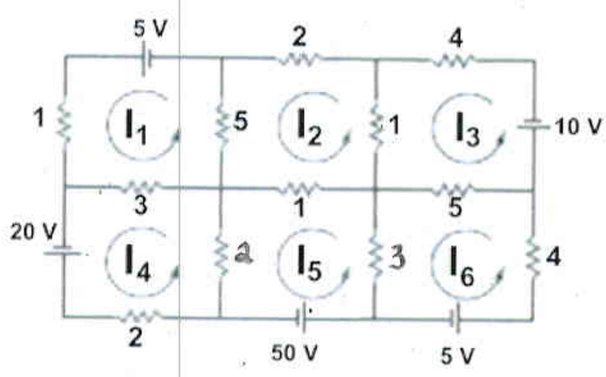
Instructions: You may [redacted] use a calculator for this part of the exam. You must show work or provide justification to receive credit (partial or otherwise) for answers. Use exact answers unless directed to round.

1. **Use an inverse matrix** to solve $\begin{cases} x_1 \quad - 3x_3 = 4 \\ -2x_1 + x_2 + x_3 = -6 \\ 2x_1 - x_2 + 4x_3 = 8 \end{cases}$. Give the inverse matrix used. You should write the matrix equation to be solved, the solution with the inverse matrix in equation form, and the final solution in vector form. (5 points)

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3/5 & 3/5 \\ 2 & 2 & 1 \\ 0 & 1/5 & 1/5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/5 & 3/5 \\ 2 & 2 & 1 \\ 0 & 1/5 & 1/5 \end{bmatrix}\begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 26/5 \\ 4 \\ 2/5 \end{bmatrix}$$

2. Solve the electrical circuit problem shown below. Write the system, and the matrix representing the system. Then solve it in your calculator. Write the solution in vector form. Round your answers to two decimal places. (8 points)



$$9I_1 - 5I_2 - 3I_4 = 5$$
$$-5I_1 + 9I_2 - I_3 - I_5 = 0$$
$$-I_2 + 10I_3 - 5I_6 = -10$$
$$-3I_1 + 7I_4 - 2I_5 = 20$$
$$-I_2 - 2I_4 + 6I_5 - 3I_6 = 50$$
$$-5I_3 - 3I_5 + 12I_6 = 5$$

$$\left[\begin{array}{cccccc|c} 9 & -5 & 0 & -3 & 0 & 0 & 5 \\ -5 & 9 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 10 & 0 & 0 & -5 & -10 \\ -3 & 0 & 0 & 7 & -2 & 0 & 20 \\ 0 & -1 & 0 & -2 & 6 & -3 & 50 \\ 0 & 0 & -5 & 0 & -3 & 12 & 5 \end{array}\right] \Rightarrow \vec{I} \approx \begin{bmatrix} 7.41 \\ 6.07 \\ 2.20 \\ 10.44 \\ 15.42 \\ 5.19 \end{bmatrix} \text{ or } \frac{1}{698}\begin{bmatrix} 5172 \\ 4240 \\ 1537 \\ 7286 \\ 10763 \\ 3622 \end{bmatrix}$$

3. The invertible matrix theorem states that several statements are equivalent to matrix A being invertible. Name 4 of these equivalent statements (so far there are 11 to choose from). (4 points)

examples include

$A\vec{x} = \vec{0}$ has only trivial solution

A is one-to-one

A is onto

A^T is invertible

4. Use the definition of a linear transformation to determine if the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - \sqrt{x_2} \\ x_3 - 2x_2 \\ x_1 + 4x_3 \end{bmatrix}$ is linear or nonlinear. You must show work to justify your answer. (5 points)

non-linear

(x_2 cannot be all real #'s to start w/)

consider $\vec{x} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ $T(\vec{x}) = \begin{bmatrix} -2 \\ -8 \\ 0 \end{bmatrix}$

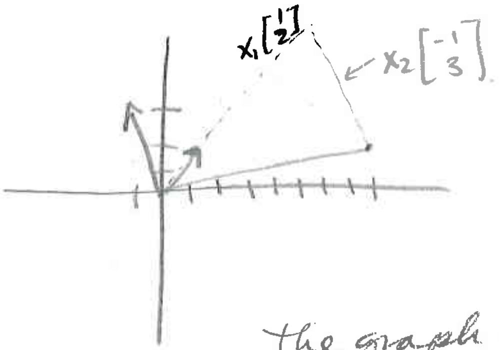
$4T(\vec{x}) = \begin{bmatrix} -8 \\ -32 \\ 0 \end{bmatrix}$ but $4\vec{x} = \begin{bmatrix} 0 \\ 16 \\ 0 \end{bmatrix}$ $T(4\vec{x}) = \begin{bmatrix} -4 \\ -32 \\ 0 \end{bmatrix}$

which are not the same vector

Should also fail addition test

5. For each of the questions below, answer as fully as possible. Justify your answer in each case.

a. The vector equation $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ is consistent and independent. Describe geometrically what that solution means. A graph would probably be helpful. (5 points)



x_1 and x_2 are the scalars & vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ respectively needed to obtain $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$

the graph suggests that x_1 is positive & x_2 is negative

b. Construct a 4x3 matrix A so that $A\vec{x} = \vec{0}$ has a non-trivial solution. (3 points)

$$A \sim \begin{bmatrix} 1 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I put my matrix in echelon form. This isn't necessary, but it must reduce so that there is a free variable in one column.

c. Explain why the serial application of three linear transformations is itself a linear transformation. (Hint: treat each transformation as a matrix; apply the transformation and then apply the next one to the result, etc. What does the resulting vector look like? What is the cumulative result of the three transformations? You may choose specific matrices to think about the problem, but your explanation should apply to generic matrices of the same size.) (5 points)

Suppose A is a linear transformation mapping $\mathbb{R}^n \to \mathbb{R}^m$ & B is a lin. trans. mapping $\mathbb{R}^m \to \mathbb{R}^p$ and C is a lin. trans mapping $\mathbb{R}^p \to \mathbb{R}^q$

$A\vec{x} = \vec{y}$ $\quad B\vec{y} = \vec{z}$ $\quad C\vec{z} = \vec{b}$ $\Rightarrow$ $C\vec{z} = C(B\vec{y}) =$

$CB(A\vec{x}) = (CBA)\vec{x} = \vec{b}$ transformation (CBA)

is $q \times n$ matrix that maps $\vec{x}$ onto $\vec{z}$. since it is a matrix it is linear

d. Explain why only square matrices can have inverses. (4 points)

for a matrix to have an inverse $A^{-1}A = I$ and $AA^{-1} = I$

version 1 this multiplication is only defined on both sides if A is Square.

alt version consider A is a $m \times n$ matrix $m \neq n$ Then A_L^{-1} would need to be $p \times m$ (or maybe $m \times m$). But A_R^{-1} would need to be $n \times p$ (or maybe $n \times n$) to be defined, but $p \times m$, & $n \times p$ are not same size since $m \neq n$. So not one A^{-1}!

6. Find the polynomial of the form $a_2t^2 + a_1t + a_0 = p(t)$ that passes through the points (2,24), (9,10), (13, -42). Write the system of equations and matrix for the system. Give the resulting polynomial. (6 points)

$$4a_2 + 2a_1 + a_0 = 24$$
$$81a_2 + 9a_1 + a_0 = 10$$
$$169a_2 + 13a_1 + a_0 = -42$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 24 \\ 81 & 9 & 1 & 10 \\ 169 & 13 & 1 & -42 \end{array}\right] \Rightarrow \text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

$$p(t) = -t^2 + 9t + 10$$