SERIES SOLUTIONS

When using analytic functions, we are constrained by the common types of functions we can assume at the start. A more general way to express functions is the power series. Since power series can be used to represent both common and uncommon functions, this gives us greater ability to find a solution to equations that don't fit any of the standard patterns we know how to solve. In addition, they give us another way to prove that the standard functions we've chosen for the common patterns does indeed work. Here, we'll consider power series solutions to simple functions (for instance, second order constant coefficient problems) that we can then check by another means. We'll also consider more general situations expanded around ordinary points and regular singular points. We'll mostly consider expanding around zero (0) since this tends to make the math easier, but we'll look at one example where we expand around another point to look at how this changes the math. We will only consider homogeneous cases.

An **ordinary point** is a point in standard form $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$ where all the $a_i(x)$ are defined.

A **regular singular point** x_0, which will only come into play later on, is a point where at least one of the $a_i(x)$ is undefined, but if we multiply by $(x - x_0)^k$ for each $i = n - k$, that these points of discontinuity disappear. In other words, multiply the first term $a_{n-1}(x)$ by $(x - x_0)$, and $a_{n-2}(x)$ by $(x - x_0)^2$, and so on.

If a point to expand the problem around is not given, I will tend to choose zero (0) for the slightly easier math, or the initial condition (with priority given to the initial condition).

In all of these ordinary point cases we will assume the solution $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, where c_n is the coefficient that goes with the nth power of x. We'll take this information and plug it into the differential equation to solve for a condition on how the coefficients relate to each other.

To plug this proposed solution into the equation we'll need to take derivatives. Each time we take a derivative, the constant disappears. In the original solution, this corresponds to n=0, so the index changes to start at n=1. Same for the next derivative.

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$
$$y' = \sum_{n=1}^{\infty} c_n n (x - x_0)^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)(x - x_0)^{n-2}$$
$$y''' = \sum_{n=3}^{\infty} c_n n(n-1)(n-2)(x - x_0)^{n-3}$$

And so forth.

A second skill we'll need is resetting the index to n=0. In order to combine the summations into a single sum, we'll need make them similar. One way to do that is make all the powers of n the same. Consider the first derivative, we want to make $(x-x_0)^{n-1}$ and change that to $(x-x_0)^n$. In the summation, we'll replace all the n's with n+1. The index of summation will then start at n=0. Consider that the first term in the original sequence, when n=1 has a power of 0, since 1-1=0. In the new sequence, where the power is n instead of n-1, we still want to get started at 0.

Thus:

$$y' = \sum_{n=1}^{\infty} c_n n(x-x_0)^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)(x-x_0)^n$$

Alternatively:

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)(x-x_0)^{n-2} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)(x-x_0)^n$$

Now, in some of our problems, we'll end up multiplying by powers of x that will change this slightly, but the principle is the same: if you bring up the index inside the sum, then the starting value has to come down by just as much.

The third thing we'll need to be able to do is to pull off initial values of a sum, when we are trying to match the starting index value of all the summations in the equation without changing the powers. We do this by remembering that

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0(x-x_0)^0 + c_1(x-x_0)^1 + c_2(x-x_0)^2 + \cdots$$

We can rewrite the sum starting at any index by breaking this into the terms before the starting index, and the rest of the sum.

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0(x-x_0)^0 + \sum_{n=1}^{\infty} c_n(x-x_0)^n = c_0(x-x_0)^0 + c_1(x-x_0)^0 + \sum_{n=2}^{\infty} c_n(x-x_0)^n$$

And so forth. Typically, we'll only have to peel off one or two terms.

Let's look at some problems.

Example 1. Use a geometric series to solve the differential equation $4y'' - y = 0$ around the point $x_0 = 0$.

Assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$. This makes the equation:

$$4\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Step 1: Bring all the coefficients inside the summations.

$$\sum_{n=2}^{\infty} 4c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Step 2: Make all your exponents the same. Usually this means x^n. The second summation is already there, but we need to adjust the index of the first sum to make this happen.

$$\sum_{n=0}^{\infty} 4c_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

Step 3: If the indices of each sum are not already the same, we'd need to make them the same. In this example, since we didn't have polynomial coefficients, just constants, adjusting the powers took care of this, too.

Step 4: Combine the sums and collect the coefficients.

$$\sum_{n=0}^{\infty} x^n[4c_{n+2}(n+2)(n+1) - c_n] = 0$$

Step 5: Set all coefficients (of each degree term and the coefficient in the sum) equal to zero, since we need the solution to equation zero for all the possible values of x.

$$4c_{n+2}(n+2)(n+1) - c_n = 0$$

We can solve for the higher index coefficient in terms of the lower indexed one.

$$c_{n+2} = \frac{c_n}{4(n+2)(n+1)}$$

This relationship means that we're going to get two solutions, one depending on n even, and one depending on n odd.

Step 6: Calculation the first several coefficients to see if you can establish a pattern.

Starting with n=0 and n=1, build a table of coefficients

n=0	$c_2 = \frac{c_0}{4(2)(1)}$	n=1	$c_3 = \frac{c_1}{4(3)(2)}$

n=2	$c_4 = \frac{c_2}{4(4)(3)} = \frac{c_0}{4^2(4)(3)(2)(1)} = \frac{c_0}{4^2 4!}$	n=3	$c_5 = \frac{c_3}{4(5)(4)} = \frac{c_1}{4^2(5)(4)(3)(2)} = \frac{c_1}{4^2 5!}$
n=4	$c_6 = \frac{c_4}{4(6)(5)} = \frac{c_0}{4^3(6)(5)4!} = \frac{c_0}{4^3 6!}$	n=5	$c_7 = \frac{c_5}{4(7)(6)} = \frac{c_1}{4^3 7!}$
n=6	$c_8 = \frac{c_6}{4(8)(7)} = \frac{c_0}{4^4 8!}$	n=7	$c_9 = \frac{c_7}{4(9)(8)} = \frac{c_1}{4^4 9!}$
n=8	$c_{10} = \frac{c_8}{4(10)(9)} = \frac{c_0}{4^5 10!}$	n=9	$c_{11} = \frac{c_9}{4(11)(10)} = \frac{c_1}{4^4 9!}$

Look at the second column. We have powers of 4, or powers of 2^2. We have twice those powers factorial. And each even coefficient depends on c_0. In the last column we have a similar situation. Powers of 4, and odd factorials, and all the coefficients depend on c_1. We can represent even numbers by 2k, and odd numbers by 2k+1, where k here starts at 0 and goes up one for each row.

Our two solutions in geometric series form are:

$$y_1 = c_0\left[1 + \frac{1}{4 \cdot 2!}x^2 + \frac{1}{4^2 4!}x^4 + \frac{1}{4^3 6!}x^6 + \frac{1}{4^4 8!}x^8 + \frac{1}{4^5 10!}x^{10} + \cdots\right] = c_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{1}{2}x\right)^{2n}$$

$$y_2 = c_1\left[x + \frac{1}{4 \cdot 3!}x^3 + \frac{1}{4^2 5!}x^5 + \frac{1}{4^3 7!}x^7 + \frac{1}{4^4 9!}x^9 + \frac{1}{4^5 11!}x^{11} + \cdots\right] = 2c_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{1}{2}x\right)^{2n+1}$$

(The extra 2 in the second equation is because I pulled the ½ inside the power; it's to cancel out the extra +1 in the exponent.)

If you consider the Taylor series formulas for some of our common functions, these geometric formulas are equivalent to $y_1 = c_0 \cos(\frac{1}{2}x)$ and $y_2 = 2c_1 \sin(\frac{1}{2}x)$.

If we solve the equation by other means, these are the answers we'd have ended up with.

So now, let's consider a slightly trickier problem: one that doesn't have an obvious closed form.

Example 2. Solve the differential equation $xy'' + y' + xy = 0$ using a series solution centered at x=0. [Note: Technically, we shouldn't solve this problem at zero because it's a singular point, not an ordinary point. When we put this equation in standard form, we get the equation is not defined at x=0. Still, this is a good example of the procedure used for the problem, and we'll see what happens at the end. We'll solve it again correctly using Frobenius' Theorem in a later example. Solving this equation at a non-singular point is also a practice problem.]

Now we have some x's involved so we are unlikely to get a nice closed form solution as we had before. Also, we will have to use all of our tricks to combine our sums.

Start with the usual $y = \sum_{n=0}^{\infty} c_n x^n$ and its derivatives and plug them into our differential equation.

$$x\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n = 0$$

Step 1: Bring any coefficients inside the summations. Here, this means the x's.

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-1} + \sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Step 2: Make all the exponents of x the same. Sometimes we might have to have starting indices starting below zero, but that's okay for now. These will get fixed in a later step, and we'll still produce a recursive algorithm. You can choose to make them all x^n or x^{n+1}. I'm going to go ahead and work with the former. In the first two sums I'm replacing n with n+1 to bring the index down and keep the power the same. In the last one, I'm replacing n with n-1 and increasing the index. Be sure to make these replacements everyone including the subscripts.

$$\sum_{n=1}^{\infty} c_{n+1} n(n+1)x^n + \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

I brought the first two sums down by one to increase the exponent, and the last index up by one to decrease the power.

Step 3: We need to make all the starting indices the same. We'll do this here by finding the highest starting index value and hold those fixed. Any sums that start at a small index, we will peel off an appropriate number of terms until we can start at the same place as the highest one. In this case, two of our summations start at 1, and one starts at 0. So the middle one needs to have the n=0 case removed. The loose term is the case where I've set n=0.

$$\sum_{n=1}^{\infty} c_{n+1} n(n+1)x^n + c_1(1)x^0 + \sum_{n=1}^{\infty} c_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

Step 4. Collect the summations together and factor out the common x^n to get a single coefficient expression. Collect any loose terms with the same power.

$$\sum_{n=1}^{\infty} x^n[c_{n+1} n(n+1) + c_{n+1}(n+1) + c_{n-1}] + c_1 = 0$$

Step 5: Set the coefficients equal to zero. This time we have two conditions.

$$c_1 = 0$$

$$c_{n+1}n(n+1) + c_{n+1}(n+1) + c_{n-1} = 0$$

Rearranging this last one we get:

$$c_{n+1}(n(n+1) + (n+1)) = -c_{n-1}$$
$$c_{n+1} = -\frac{c_{n-1}}{(n+1)^2}$$

This is still a recursive algorithm with a step of two, but it starts at n=1 just like our summation did. When n=1, we'll get c_0 on the right and c_2 on the left. However, one of our sequences, that starting at n=2, where we get c_1, the entire sequence after that will be zero. This is a terminating sequence (in this case, it terminates before it even gets started.)

Step 6: Plugging in some of these n's we get:

n=1	$c_2 = -\frac{c_0}{2^2}$	n=2	$c_3 = -\frac{c_1}{3^2} = 0$
n=3	$c_4 = -\frac{c_2}{4^2} = \frac{c_0}{2^2 4^2}$	n=4	$c_5 = -\frac{c_3}{5^2} = 0$
n=5	$c_6 = -\frac{c_4}{6^2} = -\frac{c_0}{2^2 4^2 6^2}$	n=6	$c_7 = -\frac{c_5}{7^2} = 0$

We thus have one solution which looks like:

$$y_1 = c_0\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 4^2}x^4 - \frac{1}{2^2 4^2 6^2}x^6 + \cdots\right] = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$$

The fact that we got only one solution here and not two of them is an indication that something is amiss. At a singular point, even a regular one, we are not guaranteed two solutions. We'll redo this problem later with Frobenius' Theorem to see if we can get a condition for a second solution.

Example 3. Solve the differential equation $(x+1)y'' - (2-x)y' + y = 0$ at the ordinary point x=1.

Since we know this is an ordinary point, we can proceed with our usual approach and expect two solutions. We have a bit of a problem with our coefficients, and it will be easiest to deal with those now. To bring any polynomial coefficients inside the sum, we'll need them to have the same form as the geometric term we are using, which will be (x-1). We need to rewrite our coefficients so that we have the form (x-1) and some extra constant.

$$x + 1 = (x-1) + 2$$
$$-(2-x) = x - 2 = (x-1) - 1$$

So now our equation looks like this:

$$[(x-1)+2]y'' + [(x-1)-1]y' + y = 0$$

To simplify this visually, I'm going to make the substitution $u = x - 1$. Our proposed solution will then be $y = \sum_{n=0}^{\infty} c_n (x-1)^n = \sum_{n=0}^{\infty} c_n u^n$ and our differential equation is: $(u+2)y'' + (u-1)y' + y = 0$.

Since this is merely a translation $\frac{dy}{dx} = \frac{dy}{du}$. We can solve the equation in this form, and then out it back in terms of (x-1) when we are finished. You can just work with the differential equation as we've rearranged it with x-1 still in place, but the expressions will be longer and take more paper to write out. Hopefully, this will be easier to follow with less clutter.

Let's now do the replacements for the derivatives:

$$(u+2)\sum_{n=2}^{\infty} c_n n(n-1)u^{n-2} + (u-1)\sum_{n=1}^{\infty} c_n n u^{n-1} + \sum_{n=0}^{\infty} c_n u^n = 0$$

Step 1: To combine the coefficients, we'll need to split up the parentheses.

$$u\sum_{n=2}^{\infty} c_n n(n-1)u^{n-2} + 2\sum_{n=2}^{\infty} c_n n(n-1)u^{n-2} + u\sum_{n=1}^{\infty} c_n n u^{n-1} - \sum_{n=1}^{\infty} c_n n u^{n-1} + \sum_{n=0}^{\infty} c_n u^n = 0$$

Then bring everything inside the summations.

$$\sum_{n=2}^{\infty} c_n n(n-1)u^{n-1} + \sum_{n=2}^{\infty} 2c_n n(n-1)u^{n-2} + \sum_{n=1}^{\infty} c_n n u^{n} - \sum_{n=1}^{\infty} c_n n u^{n-1} + \sum_{n=0}^{\infty} c_n u^n = 0$$

Now we have many more sums to work with, but our general procedure is exactly the same as before.

Step 2: Match the powers in each sum by adjusting the starting index in each sum.

The first sum has to come down by one since we need the power to come up by one. The second sum needs to be adjusted by 2 in the same direction. The third sum is fine as it is. The fourth sum needs to be adjusted by one just like the first two. And the last one is fine as it is.

$$\sum_{n=1}^{\infty} c_{n+1} n(n+1)u^{n} + \sum_{n=0}^{\infty} 2c_{n+2}(n+2)(n+1)u^{n} + \sum_{n=1}^{\infty} c_n n u^{n} - \sum_{n=0}^{\infty} c_{n+1}(n+1) u^{n} + \sum_{n=0}^{\infty} c_n u^n = 0$$

Step 3: All the starting indices now need to be matched. Remember, we are matching them to the highest starting index, and we'll pull out and lower terms. We have two sums starting at n=1, and three starting at n=0. These last three sums, we'll pull out the n=0 term. (Do be careful of the signs.)

$$\sum_{n=1}^{\infty} c_{n+1} n(n+1)u^n + 2c_2(2)(1)u^0 + \sum_{n=1}^{\infty} 2c_{n+2}(n+2)(n+1)u^n + \sum_{n=1}^{\infty} c_n n u^n - c_1(1)u^0 - \sum_{n=1}^{\infty} c_{n+1}(n+1)u^n + c_0 u^o + \sum_{n=1}^{\infty} c_n u^n = 0$$

Step 4: The loose terms are all constants, so they can be collected, as well as the sums now combined.

$$\sum_{n=1}^{\infty} u^n [c_{n+1} n(n+1) + 2c_{n+2}(n+2)(n+1) + c_n n - c_{n+1}(n+1) + c_n] + 4c_2 - c_1 + c_0 = 0$$

A slightly better expression can be obtained by collecting the c_i terms together.

$$\sum_{n=1}^{\infty} u^n [c_{n+1}(n-1)(n+1) + 2c_{n+2}(n+2)(n+1) + c_n(n+1)] + 4c_2 - c_1 + c_0 = 0$$

Step 5: Our coefficient relationships are then:

$$4c_2 - c_1 + c_0 = 0 \rightarrow c_2 = \frac{c_1 - c_0}{4}$$
$$c_{n+1}(n-1)(n+1) + 2c_{n+2}(n+2)(n+1) + c_n(n+1) = 0$$

Cancelling out the n+1 in the last expression gives us:

$$c_{n+1}(n-1) + 2c_{n+2}(n+2) + c_n = 0 \rightarrow c_{n+2} = \frac{-c_n - c_{n+1}(n-1)}{2}$$

In this case, we're not going to get two nice strings that depend only on every other value. Instead, we're going to get later coefficients that depend on *two* previous coefficients. But given the first relationship from the constants, we'll always be able to reduce these down to expressions that depend only on c_0 and c_1. When we separate these, we'll be able to get two solutions from them.

Step 6: We need to solve now for the first couple coefficients so that we can state the first couple terms of our solution. We won't need to solve for n=8 to get 4 terms each since we'll get terms for both series at the same time, but there may be more algebra involved to simplify.

n=0	$c_2 = \frac{c_1 - c_0}{4}$
n=1	$c_3 = \frac{-c_1 - c_2(0)}{2} = \frac{-c_1}{2}$
n=2	$c_4 = \frac{-c_2 - c_3(1)}{2} = -\frac{1}{2}\left(\frac{c_1 - c_0}{4}\right) - \frac{1}{2}\left(\frac{-c_1}{2}\right) = \frac{c_1 + c_0}{8}$
n=3	$c_5 = \frac{-c_3 - c_4(2)}{2} = -\frac{1}{2}\left(\frac{-c_1}{2}\right) - \left(\frac{c_1 + c_0}{8}\right) = \frac{c_1 - c_0}{8}$
n=4	$c_6 = \frac{-c_4 - c_5(3)}{2} = -\frac{1}{2}\left(\frac{c_1 + c_0}{8}\right) - \frac{3}{2}\left(\frac{c_1 - c_0}{8}\right) = \frac{-c_1}{4} + \frac{c_0}{8}$

To obtain our two solutions, collect the coefficients that go with c_0 together, and the same for the ones with c_1. Don't forget the term that starts the beginning of the sequence. The subscript tells you what power it goes with. Be sure to convert back to x's when you're done.

$$y_1 = c_0\left[1 - \frac{1}{4}u^2 + \frac{1}{8}u^4 - \frac{1}{8}u^5 + \frac{1}{8}u^6 + \cdots\right] =$$
$$c_0\left[1 - \frac{1}{4}(x-1)^2 + \frac{1}{8}(x-1)^4 - \frac{1}{8}(x-1)^5 + \frac{1}{8}(x-1)^6 + \cdots\right]$$

$$y_2 = c_1\left[u + \frac{1}{4}u^2 - \frac{1}{2}u^3 + \frac{1}{8}u^4 + \frac{1}{8}u^5 - \frac{1}{4}u^6 + \cdots\right] =$$
$$c_1\left[(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{2}(x-1)^3 + \frac{1}{8}(x-1)^4 + \frac{1}{8}(x-1)^5 - \frac{1}{4}(x-1)^6 + \cdots\right]$$

It doesn't look like there is a nice way to write these sums, so we'll leave them as they are.

We'll do one last example to revisit Example 2. We saw that when we expanded around x=0, which was a singular point in that equation, we only got one solution. It may be that there is only one, but let's try it again using Frobenius' Theorem to see if we can find a condition for a second solution.

Example 4. Solve the differential equation $xy'' + y' + xy = 0$ using a series solution centered at x=0. (This is a regular singular point.)

Frobenius' Theorem states that we can solve the equation at the regular singular point by using $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ instead of our regular assumption. If we can find a condition on r, we may be able to get a solution we would not have obtained otherwise. Let's try it.

$$x\sum_{n=2}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=1}^{\infty} c_n(n+r)x^{n+r-1} + x\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Step 0: Before we can proceed, we're going to factor out the x^r and discard it (since it can't be equal to zero everywhere. We now have an expression that looks very much like before, but our coefficients now have r in them. By continuing through the same steps as before, we hope to obtain a condition on r.

$$x^r\left[x\sum_{n=2}^{\infty} c_n(n+r)(n+r-1)x^{n-2} + \sum_{n=1}^{\infty} c_n(n+r)x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n = 0\right]$$

Step 1: Bring any coefficients inside the summations. Here, this means the x's.

$$\sum_{n=2}^{\infty} c_n(n+r)(n+r-1)x^{n-1} + \sum_{n=1}^{\infty} c_n(n+r)x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Step 2: Make all the exponents of x the same. Sometimes we might have to have starting indices starting below zero, but that's okay for now. These will get fixed in a later step, and we'll still produce a recursive algorithm. You can choose to make them all x^n or x^{n+1}. I'm going to go ahead and work with the former. In the first two sums I'm replacing n with n+1 to bring the index down and keep the power the same. In the last one, I'm replacing n with n-1 and increasing the index. Be sure to make these replacements everyone including the subscripts.

$$\sum_{n=1}^{\infty} c_{n+1}(n+r)(n+r+1)x^n + \sum_{n=0}^{\infty} c_{n+1}(n+r+1)x^n + \sum_{n=1}^{\infty} c_{n-1}x^n = 0$$

I brought the first two sums down by one to increase the exponent, and the last index up by one to decrease the power.

Step 3: We need to make all the starting indices the same. We'll do this here by finding the highest starting index value and hold those fixed. Any sums that start at a small index, we will peel off an appropriate number of terms until we can start at the same place as the highest one. In this case, two of our summations start at 1, and one starts at 0. So the middle one needs to have the n=0 case removed. The loose term is the case where I've set n=0.

$$\sum_{n=1}^{\infty} c_{n+1}(n+r)(n+r+1)x^n + c_1(r+1)x^0 + \sum_{n=1}^{\infty} c_{n+1}(n+r+1)x^n + \sum_{n=1}^{\infty} c_{n-1}x^n = 0$$

Step 4. Collect the summations together and factor out the common x^n to get a single coefficient expression. Collect any loose terms with the same power.

$$\sum_{n=1}^{\infty} x^n[c_{n+1}(n+r)(n+r+1) + c_{n+1}(n+r+1) + c_{n-1}] + c_1(r+1) = 0$$

Step 5: Set the coefficients equal to zero. This time we have three conditions, one of them on r.

$$c_1 = 0 \textbf{ or } r+1 = 0 \rightarrow r = -1$$

$$c_{n+1}(n+r)(n+r+1) + c_{n+1}(n+r+1) + c_{n-1} = 0$$

Rearranging this last one we get:

$$c_{n+1}(n+r)(n+r+1) + (n+r+1)) = -c_{n-1}$$
$$c_{n+1} = -\frac{c_{n-1}}{(n+r+1)^2}$$

If we let $c_1 = 0$, we get the last column of values we had before with all the coefficients equal to zero. Thus, we'll let r=(-1) And obtain the following relation:

$$c_{n+1} = -\frac{c_{n-1}}{(n-1+1)^2} = -\frac{c_{n-1}}{(n)^2}$$

Step 6: Plugging in some of these n's we get:

n=1	$c_2 = -\frac{c_0}{1^2} = -c_0$	n=2	$c_3 = -\frac{c_1}{2^2}$
n=3	$c_4 = -\frac{c_2}{3^2} = \frac{c_0}{3^2}$	n=4	$c_5 = -\frac{c_3}{4^2} = \frac{c_1}{2^2 4^2}$
n=5	$c_6 = -\frac{c_4}{5^2} = -\frac{c_0}{3^2 5^2}$	n=6	$c_7 = -\frac{c_5}{6^2} = \frac{-c_1}{2^2 4^2 6^2}$

Because of our incorrect assumption before, our answer is a little off. We do have a series of coefficients, but the powers they belong to are off by one. The two solutions we've obtained here are the correct solutions.

$$y_1 = c_0\left[1 - x^2 + \frac{1}{3^2}x^4 - \frac{1}{3^2 5^2}x^6 + \cdots\right] = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{[1 \cdot 3 \cdot 5 \cdot \ldots (2k+1)]^2}$$

$$y_2 = c_1\left[x - \frac{1}{2^2}x^3 + \frac{1}{2^2 4^2}x^5 - \frac{1}{2^2 4^2 6^2}x^7 + \cdots\right] = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k}(k!)^2}$$

It should be noted that the regular series solution case, when solved at an ordinary point, guarantees that you should obtain the correct number of solutions that matches the order of the equation: second order equations should get two solutions. However, Frobenius' Theorem is working in a domain where such solutions are not guaranteed. It may be that you obtain two solutions for a second order, or one, or possibly, if no condition on r is found, none at all.

Practice Problems. If no point is specified, center you solution around zero if zero is an ordinary point, or 1 if it is not. If neither of these points are ordinary nor a regular singular point, state that instead of solving.

a. $y'' - y = 0, x_0 = 0$
b. $y'' + 4y = 0, x_0 = 1$
c. $y'' + 4y' + 4y = 0, x_0 = 0$
d. $y'' - y = 0, x_0 = 3$
e. $y'' + xy' + 2y = 0, x_0 = 0$
f. $xy'' + y' + xy = 0, x_0 = 1$
g. $x(x+3)^2 y'' - y = 0$
h. $y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$
i. $2xy'' - y' + 2y = 0$
j. $2x^2 y'' - xy' + (x^2+1)y = 0$
k. $xy'' = xy' + y = 0$
l. $x^2 y'' - 2y = 0$
m. $x^2 y'' + 5xy' + 4y = 0$
n. $x^3 y''' - 6y = 0$
o. $(x^2+1)y'' + 2xy' = 0$
p. $(x^2+2)y'' + 3xy - y = 0, x_0 = 1$