Name KEY

Math 255, Exam #1, Summer 2012

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. For the following differential equations, determine the order of the equation, whether it is linear or non-linear, and whether it is ordinary or partial. (3 points each)
   a. $\dot{y} + ty^2 = 0$ first order, non-linear, ordinary

   b. $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = sint$ second order, linear, ordinary

   c. $y''' + ty' + (cos^2t)y = t^3$ 3rd order, linear, ordinary

   d. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 2nd order, linear, partial

   e. $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$ 2nd order, non linear, partial

2. Verify that $y = e^{3x}cos2x$ is a solution to the differential equation $y'' - 6y' + 13y = 0$. (15 points).

$$y' = 3e^{3x}\cos 2x + e^{3x}(-2)\sin 2x = e^{3x}(3\cos 2x - 2\sin 2x)$$

$$y'' = 3e^{3x}(3\cos 2x - 2\sin 2x) + e^{3x}(-6\sin 2x - 4\cos 2x)$$

$$y'' = 9e^{3x}\cos 2x - 6e^{3x}\sin 2x - 6e^{3x}\sin 2x - 4e^{3x}\cos 2x$$

$$= 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$$

$$5e^{3x}\cos 2x - 12e^{3x}\sin 2x - 6(e^{3x}(3\cos 2x - 2\sin 2x)) + 13e^{3x}\cos 2x =$$

$$5e^{3x}\cos 2x - 12e^{3x}\sin 2x - 18e^{3x}\cos 2x + 12e^{3x}\sin 2x + 13e^{3x}\cos 2x$$

$$(5 - 18 + 13)e^{3x}\cos 2x + (-12 + 12)e^{3x}\sin 2x = 0 \checkmark$$

3. $y = c_1e^x + c_2e^{-x}$ is a two-parameter solution to the differential equation $y'' - y = 0$. Find the solution to the differential equation that satisfies the initial conditions $y(0) = 0, y'(1) = e$. (10 points)

$y(0)=0$

$0 = c_1 + c_2 \qquad c_1 = -c_2 \qquad\qquad y' = c_1e^x - c_2e^{-x}$

$y'(1) = e$

$e = c_1e^1 - c_2e^{-1}$

$e = c_1e^1 + c_1e^{-1} \quad \times e$

$e^2 = c_1(e^2+1)$

$\frac{e^2}{e^2+1} = c_1 \qquad c_2 = \frac{-e^2}{e^2+1} \qquad\qquad y = \left(\frac{e^2}{e^2+1}\right)e^x + \left(\frac{-e^2}{e^2+1}\right)e^{-x}$

$y = \frac{e^{x+2} - e^{-x+2}}{e^2+1}$

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4. Find the region in the x-y plane where the differential equation $(4 - y^2)y' = x^2$ would have a unique solution whose graph passes through a point (x_0, y_0) in the region. Sketch the region. (10 points)

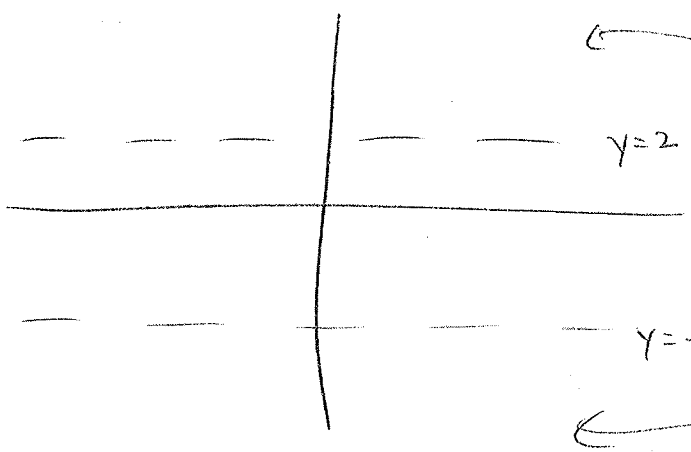
$y' = \frac{x^2}{4-y^2} \leftarrow f(x,y)$

f not continuous at $y = 2, -2$

$\frac{\partial f}{\partial y} = x^2(-1)(4-y^2)^{-2}(-2y) = \frac{2x^2y}{(4-y^2)^2}$ Same conditions

Solutions exist:

$\leftarrow$ region 3 $(2, \infty)$



$\leftarrow$ region 2 $(-2, 2)$

$\leftarrow$ region 1 $(-\infty, -2)$

5. The half-life of carbon-14 is 5730 years. Set up a differential equation and solve it to model the loss of carbon-14 in a sample over time. If there were 100mg of carbon-14 in a sample of burned wood at the beginning, and only 18mg remain now, how old is the sample? Be sure to clearly state the differential equation that models the problem. (15 points)

$$\frac{dN}{dt} = kN$$

$$N = N_0 e^{kt}$$

$$N = 100 e^{kt}$$

$$50 = 100 e^{5730k}$$

$$\frac{\ln \frac{1}{2}}{5730} = k$$

$$\frac{18}{100} = e^{\frac{\ln \frac{1}{2}}{5730} t}$$

$$\ln\left(\frac{18}{100}\right) = \frac{\ln \frac{1}{2}}{5730} t$$

$$\frac{\ln(.18)}{\ln .5} \cdot 5730 = t$$

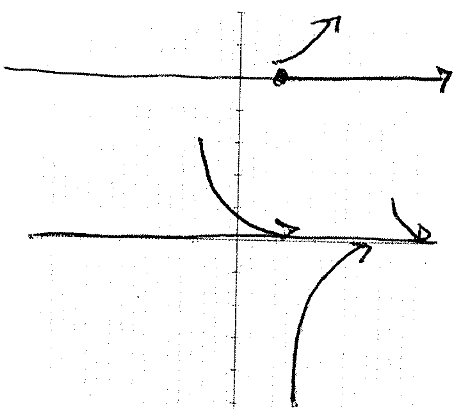
$$t = 14,175.6$$

14,176 years

6. Given the graphs below, sketch the path of a particle through the field on each graph starting at a) (1,5), b) (-1,3), c) (4,1) and d) (2,-5). The equations of the graphs are given below. Use this information to find the equilibrium points and whether they are asymptotically stable, unstable or semi-stable. (30 points)

unstable $Y=4$

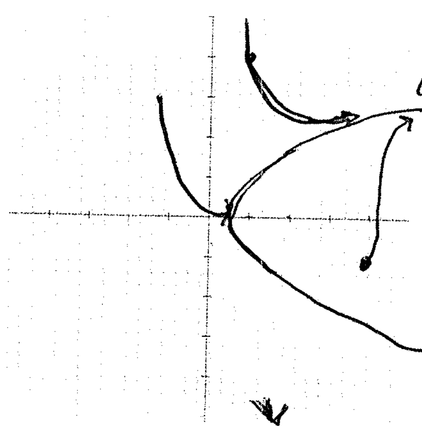
Stable $Y=0$



i.

this half stable

$X = \frac{y^2}{2} - \frac{1}{2}$



this half unstable

ii.

$$i.\, y' = y(y-4),\quad ii.\, y' = y^2 - 2x$$

7. For the following equations, determine the solution method (integrating factor/linear, separable equations, homogeneous (substitution y=vx), Bernoulli equations, exact equation) or if the problem must be done numerically. Do not solve the equations. (5 points each)

a. $y' + 3x^2y = x^2y^3$ Bernoulli (this is separable, but hard that way)

b. $y' = \frac{x^2+y^2}{2xy}$ homogeneous

c. $\sin 2x\, dx + \cos 3y\, dy = 0$ Separable

d. $y' + y = xe^{-x} + 1$ linear

e. $yy' - 2y^2 = e^x$ Bernoulli

$y' - 2y = e^x y^{-1}$

f. $xy' = (1 - y^2)^{1/2}$

Separable

g. $-y^2dx + x(x + y)dy = 0, y(1) = 1$ homogeneous

$$-2y \neq 2x+1 \quad (x^2 + xy)$$

8. Solve the differential equation $xdx + ye^{-x}\cancel{dx} = 0, y(0) = 1$ (10 points)

y

$$e^x\, ye^{-x}dy = -x\,dx\, e^x$$

$$\int y\,dy = \int -xe^x\,dx \qquad u = -x \quad dv = e^x \qquad du = -1dx \quad v = e^x$$

$$\frac{y^2}{2} = -xe^x + \int e^x\,dx$$

$$\frac{y^2}{2} = -xe^x + e^x + C$$

$$\frac{1}{2} = 0 + 1 + C$$

$$C = -\frac{1}{2}$$

$$\boxed{\frac{y^2}{2} = -xe^x + e^x - \frac{1}{2}}$$

9. Solve the linear differential equation $x^2y' + xy = 1$ either by integrating factor or by variation of parameters (integral formula). (20 points)

$$y' + \frac{1}{x}y = x^{-2}$$

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$xy' + y = \frac{1}{x}$$

$$\int (xy)' = \int \frac{1}{x}$$

$$xy = \ln x + C$$

$$\boxed{y = \frac{\ln x}{x} + \frac{C}{x}}$$

$$y = \frac{1}{x}\int x \cdot x^{-2}\,dx$$

$$y = \frac{1}{x}\int \frac{1}{x}\,dx$$

$$y = \frac{1}{x}\left[\ln x + C\right]$$

$$\boxed{y = \frac{\ln x}{x} + \frac{C}{x}}$$

10. Solve the initial value problem $(x + y)^2dx + (2xy + x^2 - 1)dy = 0, y(1) = 1$. (15 points)

$2(x+y) = 2y + 2x$ exact

$$\int (x+y)^2\,dx$$

$$\frac{1}{3}(x+y)^3 + f(y)$$

or

$$\int x^2 + 2xy + y^2\,dx$$

$$\frac{1}{3}x^3 + x^2y + xy^2 + f(y)$$

$$\int 2xy + x^2 - 1$$

$$xy^2 + x^2y - y + f(x)$$

$$\boxed{\begin{array}{c} \frac{1}{3}(x+y)^3 - y = C \\ \text{or} \\ \frac{1}{3}x^3 + x^2y + xy^2 - y = C \end{array}}$$

$\frac{1}{3}(2) - 1 = C \quad \frac{2}{3} - 1 = -\frac{1}{3} = C$

$\frac{1}{3} + 1 + \cancel{1} - \cancel{1} = C \quad \frac{4}{3} = C$

$$\frac{x^3+3xy^2}{2y^3}$$

11. Use an appropriate substitution to solve the differential equation $y' = \cancel{x^3 + 3xy^2 - 2y^2}$ (20 points)

$y = v x$

$y' = v'x + v$

$v'x + v = \frac{1+3v^2}{2v^3} - v$

$\int dv \frac{-\frac{1}{2}v^3}{-\frac{1}{2} + \frac{3}{2}v^2 + v^4} \int \frac{1}{x}\,dx = \ln x + C$

$\frac{-v^3}{v^4 - \frac{3}{2}v^2 - \frac{1}{2}} = \frac{-v^3 + v}{v^4 - \frac{3}{2}v^2 - \frac{1}{2}} \longrightarrow -\frac{1}{2}\int \frac{2v}{(v^2-\frac{3}{4})^2 - \frac{17}{16}} = -\frac{1}{2}\int \frac{du}{u^2 - a^2}$

Complete square

$\left(v^4 - \frac{3}{2}v^2 + \frac{9}{16}\right) - \frac{1}{2} - \frac{9}{16}$

$\int \frac{v^3 + v}{v^4 - \frac{3}{2}v^2 - \frac{1}{2}}\,dv$

$\ln x + C = -\frac{1}{3}\ln\left|v^4 - \frac{3}{2}v^2 - \frac{1}{2}\right| + \frac{1}{4\sqrt{17}}\ln\left|\frac{v^2 - \frac{3}{4} + \frac{\sqrt{17}}{4}}{v^2 - \frac{3}{4} - \frac{\sqrt{17}}{4}}\right|$

$\ln x + C = -\frac{1}{3}\ln\left|\frac{y^4}{x^4} - \frac{3}{2}\frac{y^2}{x^2} - \frac{1}{2}\right| + \frac{1}{4\sqrt{17}}\ln\left|\frac{\frac{y^2}{x^2} - \frac{3}{4} + \frac{\sqrt{17}}{4}}{\frac{y^2}{x^2} - \frac{3}{4} - \frac{\sqrt{17}}{4}}\right|$

12. Solve the Bernoulli equation $y' + 2xy = xy^2$ (15 points)

$1 - 2 = -1 \quad z = y^{1-n} = y^{-1} \quad z' = -y^{-2}y'$

$-y^{-2}y' - 2xy^{-1} = -x$

$z' - 2xz = -x$

$\mu = e^{\int -2x\,dx} = e^{-x^2}$

$e^{-x^2}z' - 2xe^{-x^2}z = -xe^{-x^2}$

$\int (ze^{-x^2})' = \int -xe^{-x^2}\,dx$

$e^{x^2} z e^{-x^2} = \left(+\frac{1}{2}e^{-x^2} + C\right)e^{x^2}$

$z = +\frac{1}{2} + Ce^{x^2}$

$y^{-1} = +\frac{1}{2} + Ce^{x^2}$

13. A differential equation for the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is $m\frac{dv}{dt} = mg - kv^2$ where k>0 is a constant of proportionality. The positive direction is downward.

a. Solve the equation subject to the initial condition v(0)=0. Let gravity be 32 feet/sec². (10 points)

$$v' = \frac{k}{m}\left(\frac{g}{k} - v^2\right) dt \qquad \ln\sqrt{\frac{\sqrt{\frac{g}{k}}-v}{\sqrt{\frac{g}{k}}+v}} = \frac{k}{m}t + C$$

$$\frac{dv}{\frac{g}{k}-v^2} = \frac{k}{m}dt = \frac{k}{m}x + C \qquad \frac{\sqrt{\frac{g}{k}}-v}{\sqrt{\frac{g}{k}}+v} = \left(Ae^{-\frac{k}{m}t}\right)^2$$

$$\left(\sqrt{\frac{g}{k}} - v\right)\left(\sqrt{\frac{g}{k}}+v\right)$$

$$\int \frac{1}{2}\cdot\frac{1}{\left(\sqrt{\frac{g}{k}}-v\right)} - \int\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{g}{k}}+v}\,dv = \frac{1}{2}\ln\left|\sqrt{\frac{g}{k}}-v\right| - \frac{1}{2}\ln\left|\sqrt{g/k}+v\right|$$

b. Use the solution in part a to determine the terminal velocity of the mass. (5 points)

$$\frac{\sqrt{\frac{32}{k}}-0}{\sqrt{\frac{32}{k}}+0} = 1 = \left(Ae^{\frac{k}{m}(0)}\right)^2 \qquad A = 1$$

$$\sqrt{\frac{g}{k}} = a$$

$$\frac{a-v}{a+v} = v+a\overline{)v-a} \quad -\left[1 - \frac{2a}{v+a}\right]$$

$$\frac{-v \mp a}{-2a}$$

$$\cancel{1} + \frac{2a}{v+a} = \left(Ae^{-\frac{k}{m}t}\right)^2 + 1$$

$$\frac{v+\cancel{a}}{\cancel{2a}} = \frac{2a}{1+A^2e^{-\frac{2k}{m}t}} - a$$

$$V = \frac{2\sqrt{\frac{g}{k}}}{1+A^2e^{-\frac{2k}{m}t}} - a$$

$$\to 0$$

$$\lim_{t\to\infty} \Rightarrow \frac{2\sqrt{\frac{32}{k}}}{1} - a$$