Bernoulli Equations

Bernoulli equations are first order, ordinary, nonlinear differential equations that occur in the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

when in standard form, and n is some constant.

These equations can be converted to first order, linear differential equations by means of a combination of multiplication and a substitution. The method outlined in this handout can be used whenever n≠1,0, since in both cases, simple algebra will make this a fully linear equation without the extra step of substitution.

The general procedure is as follows:

Step 1. Once the equation is in standard form, multiply the equation through by $(1-n)y^{-n}$ to collect the y's into the two terms on the left side.

Step 2. Let $z = y^{1-n}$. Take the derivative to find the rest of the substitution: $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$. This step will transform the equation into an equation linear in z.

Step 3. Solve the new equation by any method you prefer for linear equations: either by the method of integrating factors, or by using the equation for variation of parameters.

Let us do a couple examples to see how this works. We'll do Step 3 at least once each way.

Example 1. Solve the differential equation $\frac{dy}{dx} + 2xy = xy^2$.
This equation is already in standard form. So n=2. To complete step 1, we want to multiply by $(-1)y^{-2}$:

$$(-1)y^{-2}\frac{dy}{dx} + (-1)2xyy^{-2} = (-1)xy^2y^{-2}$$
$$-y^{-2}\frac{dy}{dx} - 2xy^{-1} = -x$$

This action isolates f(x) on its own on the right side.

For step 2, we make the substitution $z = y^{1-n}$, so here, $z = y^{-1}$. Taking the derivative on both sides with respect to x, we get the relationship using the chain rule:

$$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

This is the entire first term of the equation, and so we do our replacement to get:

$$\frac{dz}{dx} - 2xz = -x$$

This equation is linear in z, and so we'll do step 3 in this example by means of an integrating factor.

Find μ with the equation $\mu = e^{\int P(x)dx} = e^{\int -2xdx} = e^{-x^2}$. Multiply our z equation by this.

$$e^{-x^2}\frac{dz}{dx} - 2xe^{-x^2}z = -xe^{-x^2}$$

The left side of the equation is a product rule for $D_x\left[e^{-x^2}z\right] = e^{-x^2}\frac{dz}{dx} - 2xe^{-x^2}z$. Rewriting, we get

$$D_x\left[e^{-x^2}z\right] = -xe^{-x^2}$$

Then we integrate both sides. The right side requires u-substitution, with $u = -x^2, du = -2x$.

$$\int D_x\left[e^{-x^2}z\right]dx = \int -xe^{-x^2}\,dx$$

$$e^{-x^2}z = \frac{1}{2}e^{-x^2} + C$$

Solve for z.

$$z = \frac{1}{2}e^{-x^2}e^{x^2} + Ce^{x^2} = \frac{1}{2} + Ce^{x^2}$$

To find our implicit equation for y, then replace z with our original substitution.

$$(1-n)y^{1-n} = -(1)y^{-1} = \frac{1}{2} + Ce^{x^2}$$

If you have any initial conditions in the problem, you can solve for the constant C now.

Example 2. Solve the differential equation $y' - y = e^x\sqrt[3]{y}$.
This method works just as well for fractional exponents as it does for whole numbers. Rewrite the equation as

$$y' - y = e^x y^{1/3}$$

Multiply by $\left(1-\frac{1}{3}\right)y^{-1/3} = \frac{2}{3}y^{-1/3}$.

$$\frac{2}{3}y^{-1/3}y' - \left(\frac{2}{3}y^{-1/3}\right)y = e^x y^{1/3}\left(\frac{2}{3}y^{-1/3}\right)$$
$$\frac{2}{3}y^{-\frac{1}{3}}y' - \frac{2}{3}y^{\frac{2}{3}} = \frac{2}{3}e^x$$

For step 2, let $z = y^{1-1/3} = y^{2/3}$. For that substitution, $z' = \frac{2}{3}y^{-\frac{1}{3}}y'$ by the chain rule. So our equation becomes

$$z' - \frac{2}{3}z = \frac{2}{3}e^x$$

This equation is linear in z. To use the variation of parameters equation for the solution we have

$$z = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx = e^{2/3x}\int e^{-2/3x}\frac{2}{3}e^x dx$$
$$= \frac{2}{3}e^{2/3x}\int e^{1/3x}dx = \frac{2}{3}e^{2/3x}\left(3e^{1/3x} + C\right) = 2e^x + Ce^{2/3x}$$

I dropped the 2/3 factor in the last step by combining it with the unknown C, which you'll need initial conditions to solve for.

Which method you choose to use for step three will ultimately depend mostly on whether you prefer to memorize procedures or formulas.

Practice Problems.
Solve the Bernoulli equations.

1. $y' + 3x^2y = x^2y^3$
2. $y' + xy = xy^{-1}$
3. $y' + \frac{y}{x} = xy^2$
4. $y' + \frac{y}{x} = x\sqrt{y}$
5. $yy' - 2y^2 = e^x$
6. $y' + xy = xe^{-x^2}y^{-3}$
7. $y' + y = xy^2$
8. $\frac{dy}{dx} + 2xy = xy^2$
9. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^3}{x^2}$
10. $x\frac{dy}{dx} + y = xy^5$