9. We saw in Example 11 of Section 5.3 that there are $C(52,5)$ possible poker hands, and we assume by symmetry that they are all equally likely. In order to solve this problem, we need to compute the number of poker hands that do not contain the queen of hearts. Such a hand is simply an unordered selection from a deck with 51 cards in it (all cards except the queen of hearts), so there are $C(51,5)$ such hands. Therefore the answer to the question is the ratio

$$\frac{C(51,5)}{C(52,5)} = \frac{47}{52} \approx 90.4\% .$$

15. We need to compute the number of ways to hold two pairs. To specify the hand we first choose the kinds (ranks) the pairs will be (such as kings and fives); there are $C(13,2) = 78$ ways to do this, since we need to choose 2 kinds from the 13 possible kinds. Then we need to decide which 2 cards of each of the kinds of the pairs we want to include. There are 4 cards of each kind (4 suits), so there are $C(4,2) = 6$ ways to make each of these two choices. Finally, we need to decide which card to choose for the fifth card in the hand. We cannot choose any card in either of the 2 kinds that are already represented (we do not want to construct a full house by accident), so there are $52 - 8 = 44$ cards to choose from and hence $C(44,1) = 44$ ways to make the choice. Putting this all together by the product rule, there are $78 \cdot 6 \cdot 6 \cdot 44 = 123{,}552$ different hands classified as "two pairs."

Since each hand is equally likely, and since there are $C(52,5) = 2{,}598{,}960$ different hands (see Example 11 in Section 5.3), the probability of holding two pairs is $123552/2598960 = 198/4165 \approx 0.0475$.

17. First we need to compute the number of ways to hold a straight. We can specify the hand by first choosing the starting (lowest) kind for the straight. Since the straight can start with any card from the set $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, there are $C(10,1) = 10$ ways to do this. Then we need to decide which card of each of the kinds in the straight we want to include. There are 4 cards of each kind (4 suits), so there are $C(4,1) = 4$ ways to make each of these 5 choices. Putting this all together by the product rule, there are $10 \cdot 4^5 = 10{,}240$ different hands containing a straight. (For poker buffs, it should be pointed out that a hand is classified as a "straight" in poker if it contains a straight but does not contain a straight flush, which is a straight in which all of the cards are in the same suit. Since there are $10 \cdot 4 = 40$ straight flushes, we would need to subtract 40 from our answer above in order to find the number of hands classified as a "straight." Also, some poker books do not count $A, 2, 3, 4, 5$ as a straight.)

Since each hand is equally likely, and since there are $C(52,5) = 2{,}598{,}960$ different hands (see Example 11 in Section 5.3), the probability of holding a hand containing a straight is $10240/2598960 = 128/32487 \approx 0.00394$.

21. Looked at properly, this is the same as Exercise 7. There are 2 equally likely outcomes for the parity on the roll of a die—even and odd. Of the $2^6 = 64$ parity outcomes in the roll of a die 6 times, only one consists of 6 odd numbers. Therefore the probability is $1/64$.

29. There is only one winning choice of numbers, namely the same 8 numbers the computer chooses. Therefore the probability of winning is $1/C(100,8) \approx 1/(1.86 \times 10^{11})$.

35. **a)** There are 18 red numbers and 38 numbers in all, so the probability is $18/38 = 9/19 \approx 0.474$.
b) There are 38^2 equally likely outcomes for two spins, since each spin can result in 38 different outcomes. Of these, 18^2 are a pair of black numbers. Therefore the probability is $18^2/38^2 = 81/361 \approx 0.224$.
c) There are 2 outcomes being considered here, so the probability is $2/38 = 1/19$.
d) There are 38^5 equally likely outcomes in five spins of the wheel. Since 36 outcomes on each spin are not 0 or 00, there are 36^5 outcomes being considered. Therefore the probability is $36^5/38^5 = 1889568/2476099 \approx 0.763$.

41. **a)** There are 6^4 possible outcomes when a die is rolled four times. There are 5^4 outcomes in which a 6 does not appear, so the probability of not rolling a 6 is $5^4/6^4$. Therefore the probability that at least one 6 does appear is $1 - 5^4/6^4 = 671/1296$, which is about 0.518.

b) There are 36^{24} possible outcomes when a pair of dice is rolled 24 times. There are 35^{24} outcomes in which a double 6 does not appear, so the probability of not rolling a double 6 is $35^{24}/36^{24}$. Therefore the probability that at least one double 6 does appear is $1-35^{24}/36^{24}$, which is about 0.491. No, the probability is not greater than 1/2.

c) From our answers above we see that the answer is yes, since $0.518 > 0.491$.

1. We are told that $p(H) = 3p(T)$. We also know that $p(H) + p(T) = 1$, since heads and tails are the only two outcomes. Solving these simultaneous equations, we find that $p(T) = 1/4$ and $p(H) = 3/4$. An interesting example of an experiment in which intuition would tell you that outcomes should be equally likely but in fact they are not is to spin a penny on its edge on a smooth table and let it fall. Repeat this experiment 50 times or so, and you will be amazed at the outcomes. Make sure to count only those trials in which the coin spins freely for a second or more, not bumping into any objects or falling off the table.

5. There are six ways to roll a sum of 7. We can denote them as $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$, where (i,j) means rolling i on the first die and j on the second. We need to compute the probability of each of these outcomes and then add them to find the probability of rolling a 7. The two dice are independent, so we can argue as follows, using the given information about the probability of each outcome on each die: $p((1,6)) = \frac{1}{7}\cdot\frac{1}{7} = \frac{1}{49}$; $p((2,5)) = \frac{1}{7}\cdot\frac{1}{7} = \frac{1}{49}$; $p((3,4)) = \frac{1}{7}\cdot\frac{1}{7} = \frac{1}{49}$; $p((4,3)) = \frac{2}{7}\cdot\frac{2}{7} = \frac{4}{49}$; $p((5,2)) = \frac{1}{7}\cdot\frac{1}{7} = \frac{1}{49}$; $p((6,1)) = \frac{1}{7}\cdot\frac{1}{7} = \frac{1}{49}$. Adding, we find that the probability of rolling a 7 as the sum is $9/49$.

7. We exploit symmetry in answering many of these.

a) Since 1 has either to precede 4 or to follow it, and there is no reason that one of these should be any more likely than the other, we immediately see that the answer is 1/2. We could also use brute force here, list all 24 permutations, and count that 12 of them have 1 preceding 4.

b) By the same reasoning as in part **(a)**, the answer is again 1/2.

c) We could list all 24 permutations, and count that 8 of them have 4 preceding both 1 and 2. But here is a better argument. Among the numbers 1, 2, and 4, each is just as likely as the others to occur first. Thus by symmetry the answer is 1/3.

d) We could list all 24 permutations, and count that 6 of them have 4 preceding 1, 2, and 3 (i.e., 4 occurring first); or we could argue that there are $3! = 6$ ways to write down the rest of a permutation beginning with 4. But here is a better argument. Each of the four numbers is just as likely as the others to occur first. Thus by symmetry the answer is 1/4.

e) We could list all 24 permutations, and count that 6 of them have 4 preceding 3, and 2 preceding 1. But here is a better argument. Between 4 and 3, each is just as likely to precede the other, so the probability that 4 precedes 3 is 1/2. Similarly, the probability that 2 precedes 1 is 1/2. The relative position of 4 and 3 is independent of the relative position of 2 and 1, so the probability that both happen is the product $(1/2)(1/2) = 1/4$.

11. Clearly $p(E \cup F) \geq p(E) = 0.7$. Also, $p(E \cup F) \leq 1$. If we apply Theorem 2 from Section 6.1, we can rewrite this as $p(E) + p(F) - p(E \cap F) \leq 1$, or $0.7 + 0.5 - p(E \cap F) \leq 1$. Solving for $p(E \cap F)$ gives $p(E \cap F) \geq 0.2$.

19. As instructed, we are assuming here that births are independent and the probability of a birth in each month is 1/12. Although this is clearly not exactly true (for example, the months do not all have the same lengths), it is probably close enough for our answers to be approximately accurate.

a) The probability that the second person has the same birth month as the first person (whatever that was) is 1/12.

b) We proceed as in Example 13. The probability that all the birth months are different is

$$p_n = \frac{11}{12} \cdot \frac{10}{12} \cdots \frac{13-n}{12}$$

since each person after the first must have a different birth month from all the previous people in the group. Note that if $n \geq 13$, then $p_n = 0$ since the 12^{th} fraction is 0 (this also follows from the pigeonhole principle). The probability that at least two are born in the same month is therefore $1 - p_n$.

c) We compute $1 - p_n$ for $n = 2, 3, \ldots$ and find that the first time this exceeds $1/2$ is when $n = 5$, so that is our answer. With five people, the probability that at least two will share a birth month is about 62%.

21. If n people are chosen at random, then the probability that all of them were born on a day other than April 1 is $(365/366)^n$. To compute the probability that exactly one of them is born on April 1, we note that this can happen in n different ways (it can be any of the n people), and the probability that it happens for each particular person is $(1/366)(365/366)^{n-1}$, since the other $n-1$ people must be born on some other day. Putting this all together, the probability that two of them were born on April 1 is $1 - (365/366)^n - n(1/366)(365/366)^{n-1}$. Using a calculator or computer algebra system, we find that this first exceeds $1/2$ when $n = 614$. Interestingly, if the problem asked about *exactly* two April 1 birthdays, then the probability is $C(n,2)(1/366)^2(365/366)^{n-2}$, which never exceeds $1/2$.

27. In each case we need to compute $p(E)$, $p(F)$, and $p(E \cap F)$. Then we need to compare $p(E) \cdot p(F)$ to $p(E \cap F)$; if they are equal, then by definition the events are independent, and otherwise they are not. We assume that boys and girls are equally likely, and that successive births are independent. (Medical science suggests that neither of these assumptions is exactly correct, although both are reasonably good approximations.)

a) If the family has only two children, then there are four equally likely outcomes: BB, BG, GB, and GG. There are two ways to have children of both sexes, so $p(E) = 2/4$. There are three ways to have at most one boy, so $p(F) = 3/4$. There are two ways to have children of both sexes and at most one boy, so $p(E \cap F) = 2/4$. Since $p(E) \cdot p(F) = 3/8 \neq 2/4$, the events are not independent.

b) If the family has four children, then there are 16 equally likely outcomes, since there are 16 strings of length 4 consisting of B's and G's. All but two of these outcomes give children of both sexes, so $p(E) = 14/16$. Only five of them result in at most one boy, so $p(F) = 5/16$. There are four ways to have children of both sexes and at most one boy, so $p(E \cap F) = 4/16$. Since $p(E) \cdot p(F) = 35/128 \neq 4/16$, the events are not independent.

c) If the family has five children, then there are 32 equally likely outcomes, since there are 32 strings of length 5 consisting of B's and G's. All but two of these outcomes give children of both sexes, so $p(E) = 30/32$. Only six of them result in at most one boy, so $p(F) = 6/32$. There are five ways to have children of both sexes and at most one boy, so $p(E \cap F) = 5/32$. Since $p(E) \cdot p(F) = 45/256 \neq 5/32$, the events are not independent.

34. We need to use the binomial distribution, which tells us that the probability of k successes is

$$b(k; n, p) = C(n,k)p^k(1-p)^{n-k}.$$

a) Here $k = 0$, since we want all the trials to result in failure. Plugging in and computing, we have $b(0; n, p) = 1 \cdot p^0 \cdot (1-p)^n = (1-p)^n$.

b) There is at least one success if and only if it is not the case that there are no successes. Thus we obtain the answer by subtracting the probability in part (a) from 1, namely $1 - (1-p)^n$.

c) There are two ways in which there can be at most one success: no successes or one success. We already computed that the probability of no successes is $(1-p)^n$. Plugging in $k = 1$, we compute that the probability of exactly one success is $b(1; n, p) = n \cdot p^1 \cdot (1-p)^{n-1}$. Therefore the answer is $(1-p)^n + np(1-p)^{n-1}$. This formula only makes sense if $n > 0$, of course; if $n = 0$, then the answer is clearly 1.

d) Since this event is just that the event in part (c) does not happen, the answer is $1 - [(1-p)^n + np(1-p)^{n-1}]$. Again, this is for $n > 0$; the probability is clearly 0 if $n = 0$.

4. This is identical to Exercise 2, except that $p = 0.6$. Thus the expected number of heads is $10 \cdot 0.6 = 6$.

6. There are $C(50, 6)$ equally likely possible outcomes when the state picks its winning numbers. The probability of winning \$10 million is therefore $1/C(50, 6)$, and the probability of winning \$0 is $1 - (1/C(50, 6))$. By definition, the expectation is therefore $\$10{,}000{,}000 \cdot 1/C(50, 6) + 0 = \$10{,}000{,}000/15{,}890{,}700 \approx \$0.63$.

7. By Theorem 3 we know that the expectation of a sum is the sum of the expectations. In the current exercise we can let X be the random variable giving the score on the true-false questions and let Y be the random variable giving the score on the multiple choice questions. In order to compute the expectation of X and of Y, let us for a moment ignore the point values, and instead just look at the number of true-false or multiple choice questions that Linda gets right. The expected number of true-false questions she gets right is the expectation of the number of successes when 50 Bernoulli trials are performed with $p = 0.9$. By Theorem 2 the expectation for the number of successes is $np = 50 \cdot 0.9 = 45$. Since each problem counts 2 points, the expectation of X is $45 \cdot 2 = 90$. Similarly, the expected number of multiple choice questions she gets right is the expectation of the number of successes when 25 Bernoulli trials are performed with $p = 0.8$, namely $25 \cdot 0.8 = 20$. Since each problem counts 4 points, the expectation of Y is $20 \cdot 4 = 80$. Therefore her expected score on the exam is $E(X + Y) = E(X) + E(Y) = 90 + 80 = 170$.

17. The random variable that counts the number of integers we need to select has a geometric distribution with $p = 1/2302$. According to Theorem 4 the expected value is $1/(1/2302) = 2302$.