Math 266 Summer 2010
Homework #6

3. **a)** There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains two pigeons. By the pigeonhole principle the answer is 3. If three socks are taken from the drawer, at least two must have the same color. On the other hand two socks are not enough, because one might be brown and the other black. Note that the number of socks was irrelevant (assuming that it was at least 3).

b) He needs to take out 14 socks in order to insure at least two black socks. If he does so, then at most 12 of them are brown, so at least two are black. On the other hand, if he removes 13 or fewer socks, then 12 of them could be brown, and he might not get his pair of black socks. This time the number of socks did matter.

5. There are four possible remainders when an integer is divided by 4 (these are the pigeonholes here): 0, 1, 2, or 3. Therefore, by the pigeonhole principle at least two of the five given remainders (these are the pigeons) must the same.

13. **a)** We can group the first eight positive integers into four subsets of two integers each, each subset adding up to 9: $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, and $\{4,5\}$. If we select five integers from this set, then by the pigeonhole principle (at least) two of them must come from the same subset. These two integers have a sum of 9, as desired.

b) No. If we select one element from each of the subsets specified in part **(a)**, then no sum will be 9. For example, we can select 1, 2, 3, and 4.

3. If we want the permutation to end with a, then we may as well forget about the a, and just count the number of permutations of $\{b, c, d, e, f, g\}$. Each permutation of these 6 letters, followed by a, will be a permutation of the desired type, and conversely. Therefore the answer is $P(6,6) = 6! = 720$.

19. **a)** Each flip can be either heads or tails, so there are $2^{10} = 1024$ possible outcomes.

b) To specify an outcome that has exactly two heads, we simply need to choose the two flips that came up heads. There are $C(10,2) = 45$ such outcomes.

c) To contain at most three tails means to contain three tails, two tails, one tail, or no tails. Reasoning as in part **(b)**, we see that there are $C(10,3) + C(10,2) + C(10,1) + C(10,0) = 120 + 45 + 10 + 1 = 176$ such outcomes.

d) To have an equal number of heads and tails in this case means to have five heads. Therefore the answer is $C(10,5) = 252$.

21. **a)** If BCD is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of five items—the letters A, E, F, and G, and the superletter BCD. Therefore the answer is $P(5,5) = 5! = 120$.

b) Reasoning as in part **(a)**, we see that the answer is $P(4,4) = 4! = 24$.

25. a) Since the prizes are different, we want an ordered arrangement of four numbers from the set of the first 100 positive integers. Thus there are $P(100,4) = 94{,}109{,}400$ ways to award the prizes.

b) If the grand prize winner is specified, then we need to choose an ordered set of three tickets to win the other three prizes. This can be done is $P(99,3) = 941{,}094$ ways.

c) We can first determine which prize the person holding ticket 47 will win (this can be done in 4 ways), and then we can determine the winners of the other three prizes, exactly as in part **(b)**. Therefore the answer is $4P(99,3) = 3{,}764{,}376$.

d) This is the same calculation as in part **(a)**, except that there are only 99 viable tickets. Therefore the answer is $P(99,4) = 90{,}345{,}024$. Note that this answer plus the answer to part **(c)** equals the answer to part **(a)**, since the person holding ticket 47 either wins the grand prize or does not win the grand prize.

e) This is similar to part **(c)**. There are $4 \cdot 3 = 12$ ways to determine which prizes these two lucky people will win, after which there are $P(98,2) = 9506$ ways to award the other two prizes. Therefore the answer is $12 \cdot 9506 = 114{,}072$.

f) This is like part **(e)**. There are $P(4,3) = 24$ ways to choose the prizes for the three people mentioned, and then 97 ways to choose the other winner. This gives $24 \cdot 97 = 2328$ ways in all.

g) Here it is just a matter of ordering the prizes for these four people, so the answer is $P(4,4) = 24$.

h) This is similar to part **(d)**, except that this time the pool of viable numbers has only 96 numbers in it. Therefore the answer is $P(96,4) = 79{,}727{,}040$.

i) There are four ways to determine the grand prize winner under these conditions. Then there are $P(99,3)$ ways to award the remaining prizes. This gives an answer of $4P(99,3) = 3{,}764{,}376$.

j) First we need to choose the prizes for the holder of 19 and 47. Since there are four prizes, there are $P(4,2) = 12$ ways to do this. Then there are 96 people who might win the remaining prizes, and there are $P(96,2) = 9120$ ways to award these prizes. Therefore the answer is $12 \cdot 9120 = 109{,}440$.

31. We need to be careful here, because strings can have repeated letters.

a) We need to choose the position for the vowel, and this can be done in 6 ways. Next we need to choose the vowel to use, and this can be done in 5 ways. Each of the other five positions in the string can contain any of the 21 consonants, so there are 21^5 ways to fill the rest of the string. Therefore the answer is $6 \cdot 5 \cdot 21^5 = 122{,}523{,}030$.

b) We need to choose the position for the vowels, and this can be done in $C(6,2) = 15$ ways (we need to choose two positions out of six). We need to choose the two vowels (5^2 ways). Each of the other four positions in the string can contain any of the 21 consonants, so there are 21^4 ways to fill the rest of the string. Therefore the answer is $15 \cdot 5^2 \cdot 21^4 = 72{,}930{,}375$.

c) The best way to do this is to count the number of strings with no vowels and subtract this from the total number of strings. We obtain $26^6 - 21^6 = 223{,}149{,}655$.

d) As in part **(c)**, we will do this by subtracting from the total number of strings, the number of strings with no vowels and the number of strings with one vowel (this latter quantity having been computed in part **(a)**). We obtain $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 = 223149655 - 122523030 = 100{,}626{,}625$.

33. We are told that we must select three of the 10 men and three of the 15 women. This can be done is $C(10,3)C(15,3) = 54{,}600$ ways.

41. If there are no ties, then there are $3! = 6$ possible finishes. If two of the horses tie and the third has a different time, then there are 3 ways to decide which horse is not tied and then 2 ways to decide whether that horse finishes first or last. That gives $3 \cdot 2 = 6$ possibilities. Finally, all three horses can tie. So the answer is $6 + 6 + 1 = 13$.