Math 266 Summer 2010
Homework #5

3. **a)** Plugging in $n = 1$ we have that $P(1)$ is the statement $1^2 = 1 \cdot 2 \cdot 3/6$.

b) Both sides of $P(1)$ shown in part **(a)** equal 1.

c) The inductive hypothesis is the statement that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

d) For the inductive step, we want to show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can show

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

e) The left-hand side of the equation in part **(d)** equals, by the inductive hypothesis, $k(k+1)(2k+1)/6 + (k+1)^2$. We need only do a bit of algebraic manipulation to get this expression into the desired form: factor out $(k+1)/6$ and then factor the rest. In detail,

$$\begin{aligned}(1^2 + 2^2 + \cdots + k^2) + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)}\\ &= \frac{k+1}{6}\bigl(k(2k+1) + 6(k+1)\bigr) = \frac{k+1}{6}(2k^2 + 7k + 6)\\ &= \frac{k+1}{6}(k+2)(2k+3) = \frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

11. **a)** Let us compute the values of this sum for $n \leq 4$ to see whether we can discover a pattern. For $n = 1$ the sum is $\frac{1}{2}$. For $n = 2$ the sum is $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. For $n = 3$ the sum is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$. And for $n = 4$ the sum is $15/16$. The pattern seems pretty clear, so we conjecture that the sum is always $(2^n - 1)/2^n$.

b) We have already verified that this is true in the base case (in fact, in four base cases). So let us assume it for k and try to prove it for $k + 1$. More formally, we are letting $P(n)$ be the *statement* that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n},$$

and trying to prove that $P(n)$ is true for all n. We have already verified $P(1)$ (as well as $P(2)$, $P(3)$, and $P(4)$ for good measure). We now assume the inductive hypothesis $P(k)$, which is the equation displayed above with k substituted for n, and must derive $P(k+1)$, which is the equation

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}.$$

The "obvious" thing to try is to add $1/2^{k+1}$ to both sides of the inductive hypothesis and see whether the algebra works out as we hope it will. We obtain

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} = \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} = \frac{2 \cdot 2^k - 2 \cdot 1 + 1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}},$$

as desired.

17. This proof follows the basic pattern of the solution to Exercise 3, but the algebra gets more complex. The statement $P(n)$ that we wish to prove is

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30},$$

where n is a positive integer. The basis step, $n = 1$, is true, since $1 \cdot 2 \cdot 3 \cdot 5/30 = 1$. Assume the displayed statement as the inductive hypothesis, and proceed as follows to prove $P(n+1)$:

$$\begin{aligned}(1^4 + 2^4 + \cdots + n^4) + (n+1)^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + (n+1)^4 \\ &= \frac{n+1}{30}\left(n(2n+1)(3n^2+3n-1) + 30(n+1)^3\right) \\ &= \frac{n+1}{30}(6n^4 + 39n^3 + 91n^2 + 89n + 30) \\ &= \frac{n+1}{30}(n+2)(2n+3)(3(n+1)^2 + 3(n+1) - 1)\end{aligned}$$

The last equality is straightforward to check; it was obtained not by attempting to factor the next to last expression from scratch but rather by knowing exactly what we expected the simplified expression to be.

18. a) Plugging in $n = 2$, we see that $P(2)$ is the statement $2! < 2^2$.

b) Since $2! = 2$, this is the true statement $2 < 4$.

c) The inductive hypothesis is the statement that $k! < k^k$.

d) For the inductive step, we want to show for each $k \geq 2$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can prove that $(k+1)! < (k+1)^{k+1}$.

e) $(k+1)! = (k+1)k! < (k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction the statement is true for every positive integer n greater than 1.

33. To prove that $P(n): 5 \mid (n^5 - n)$ holds for all nonnegative integers n, we first check that $P(0)$ is true; indeed $5 \mid 0$. Next assume that $5 \mid (n^5 - n)$, so that we can write $n^5 - n = 5t$ for some integer t. Then we want to prove $P(n+1)$, namely that $5 \mid ((n+1)^5 - (n+1))$. We expand and then factor the right-hand side to obtain

$$\begin{aligned}(n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 \\ &= (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n) \\ &= 5t + 5(n^4 + 2n^3 + 2n^2 + n) \quad \text{(by the inductive hypothesis)} \\ &= 5(t + n^4 + 2n^3 + 2n^2 + n).\end{aligned}$$

Thus we have shown that $(n+1)^5 - (n+1)$ is also a multiple of 5, and our proof by induction is complete. (Note that here we have used n as the dummy variable in the inductive step, rather than k. It really makes no difference.)

We should point out that using mathematical induction is not the only way to prove this proposition; it can also be proved by considering the five cases determined by the value of n **mod** 5. The reader is encouraged to write down such a proof.

47. The one and only flaw in this proof is in this statement, which is part of the inductive step: "the set of the first n horses and the set of the last n horses [in the collection of $n+1$ horses being considered] overlap." The only assumption made about the number n in this argument is that n is a positive integer. When $n = 1$, so that $n + 1 = 2$, the statement quoted is obviously nonsense: the set of the first one horse and the set of the last one horse, in this set of two horses, are disjoint.

3. In each case we compute the subsequent terms by plugging into the recursive formula, using the previously given or computed values.

a) $f(2) = f(1) + 3f(0) = 2 + 3(-1) = -1$; $f(3) = f(2) + 3f(1) = -1 + 3 \cdot 2 = 5$; $f(4) = f(3) + 3f(2) = 5 + 3(-1) = 2$; $f(5) = f(4) + 3f(3) = 2 + 3 \cdot 5 = 17$

b) $f(2) = f(1)^2 f(0) = 2^2 \cdot (-1) = -4$; $f(3) = f(2)^2 f(1) = (-4)^2 \cdot 2 = 32$; $f(4) = f(3)^2 f(2) = 32^2 \cdot (-4) = -4096$; $f(5) = f(4)^2 f(3) = (-4096)^2 \cdot 32 = 536{,}870{,}912$

c) $f(2) = 3f(1)^2 - 4f(0)^2 = 3 \cdot 2^2 - 4 \cdot (-1)^2 = 8$; $f(3) = 3f(2)^2 - 4f(1)^2 = 3 \cdot 8^2 - 4 \cdot 2^2 = 176$; $f(4) = 3f(3)^2 - 4f(2)^2 = 3 \cdot 176^2 - 4 \cdot 8^2 = 92{,}672$; $f(5) = 3f(4)^2 - 4f(3)^2 = 3 \cdot 92672^2 - 4 \cdot 176^2 = 25{,}764{,}174{,}848$

d) $f(2) = f(0)/f(1) = (-1)/2 = -1/2$; $f(3) = f(1)/f(2) = 2/(-\frac{1}{2}) = -4$; $f(4) = f(2)/f(3) = (-\frac{1}{2})/(-4) = 1/8$; $f(5) = f(3)/f(4) = (-4)/\frac{1}{8} = -32$

7. There are many correct answers for these sequences. We will give what we consider to be the simplest ones.

a) Clearly each term in this sequence is 6 greater than the preceding term. Thus we can define the sequence by setting $a_1 = 6$ and declaring that $a_{n+1} = a_n + 6$ for all $n \geq 1$.

b) This is just like part (a), in that each term is 2 more than its predecessor. Thus we have $a_1 = 3$ and $a_{n+1} = a_n + 2$ for all $n \geq 1$.

c) Each term is 10 times its predecessor. Thus we have $a_1 = 10$ and $a_{n+1} = 10a_n$ for all $n \geq 1$.

d) Just set $a_1 = 5$ and declare that $a_{n+1} = a_n$ for all $n \geq 1$.

12. The basis step ($n = 1$) is clear, since $f_1^2 = f_1 f_2 = 1$. Assume the inductive hypothesis. Then $f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2}$, as desired.

7. Three-letter initials are determined by specifying the first initial (26 ways), then the second initial (26 ways), and then the third initial (26 ways). Therefore by the product rule there are $26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$ possible three-letter initials.

17. The easiest way to count this is to find the total number of ASCII strings of length five and then subtract off the number of such strings that do not contain the @ character. Since there are 128 characters to choose from in each location in the string, the answer is $128^5 - 127^5 = 34{,}359{,}738{,}368 - 33{,}038{,}369{,}407 = 1{,}321{,}368{,}961$.

26. $10^3 26^3 + 26^3 10^3 = 35{,}152{,}000$

33. For each part of this problem, we need to find the number of one-to-one functions from a set with 5 elements to a set with k elements. To specify such a function, we need to make 5 choices, in succession, namely the values of the function at each of the 5 elements in its domain. Therefore the product rule applies. The first choice can be made in k ways, since any element of the codomain can be the image of the first element of the domain. After that choice has been made, there are only $k - 1$ elements of the codomain available to be the image of the second element of the domain, since images must be distinct for the function to be one-to-one. Similarly, for the third element of the domain, there are $k - 2$ possible choices for a function value. Continuing in this way, and applying the product rule, we see that there are $k(k-1)(k-2)(k-3)(k-4)$ one-to-one functions from a set with 5 elements to a set with k elements.

a) By the analysis above the answer is $4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 = 0$, what we would expect since there are no one-to-one functions from a set to a strictly smaller set.

b) By the analysis above the answer is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

c) By the analysis above the answer is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$.

d) By the analysis above the answer is $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$.

41. a) Here is a good way (but certainly not the only way) to approach this problem. Since the bride and groom must stand next to each other, let us treat them as one unit. Then the question asks for the number of ways to arrange five units in a row (the bride-and-groom unit and the four other people). We can think of filling five positions one at a time, so by the product rule there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to make these choices. This is not quite the answer, however, since there are also two ways to decide on which side of the groom the bride will stand. Therefore the final answer is $120 \cdot 2 = 240$.

b) There are clearly $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ arrangements in all. We just determined in part **(a)** that 240 of them involve the bride standing next to the groom. Therefore there are $720 - 240 = 480$ ways to arrange the people with the bride not standing next to the groom.

c) Of the 720 arrangements of these people (see part **(b)**), exactly half must have the bride somewhere to the left of the groom. (We are invoking **symmetry** here—a useful tool for solving some combinatorial problems.) Therefore the answer is $720/2 = 360$.