Math 266 Summer 2010
Homework #4

1. **a)** The expression $1/x$ is meaningless for $x = 0$, which is one of the elements in the domain; thus the "rule" is no rule at all. In other words, $f(0)$ is not defined.
b) Things like $\sqrt{-3}$ are undefined (or, at best, are complex numbers).
c) The "rule" for f is ambiguous. We must have $f(x)$ defined uniquely, but here there are two values associated with every x, the positive square root and the negative square root of $x^2 + 1$.

7. In each case, the domain is the set of possible inputs for which the function is defined, and the range is the set of all possible outputs on these inputs.
a) The domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$, since we are told that the function operates on pairs of positive integers (the word "pair" in mathematics is usually understood to mean ordered pair). Since the maximum is again a positive integer, and all positive integers are possible maximums (by letting the two elements of the pair be the same), the range is $\mathbf{Z}^+$.

b) We are told that the domain is $\mathbf{Z}^+$. Since the decimal representation of an integer has to have at least one digit, at most nine digits do not appear, and of course the number of missing digits could be any number less than 9. Thus the range is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
c) We are told that the domain is the set of bit strings. The block 11 could appear no times, or it could appear any positive number of times, so the range is $\mathbf{N}$.
d) We are told that the domain is the set of bit strings. Since the first 1 can be anywhere in the string, its position can be $1, 2, 3, \ldots$. If the bit string contains no 1's, the value is 0 by definition. Therefore the range is $\mathbf{N}$.

10. **a)** This is one-to-one. **b)** This is not one-to-one, since b is the image of both a and b.
c) This is not one-to-one, since d is the image of both a and d.

12. **a)** This is one-to-one, since if $n_1 - 1 = n_2 - 1$, then $n_1 = n_2$.
b) This is not one-to-one, since, for example, $f(3) = f(-3) = 10$.
c) This is one-to-one, since if $n_1^3 = n_2^3$, then $n_1 = n_2$ (take the cube root of each side).
d) This is not one-to-one, since, for example, $f(3) = f(4) = 2$.

25. The function is not one-to-one (for example, $f(2) = 2 = f(-2)$), so it is not invertible. On the restricted domain, the function is the identity function from the set of nonnegative real numbers to itself, $f(x) = x$, so it is one-to-one and onto and therefore invertible; in fact, it is its own inverse.

1. **a)** $a_0 = 2 \cdot (-3)^0 + 5^0 = 2 \cdot 1 + 1 = 3$ **b)** $a_1 = 2 \cdot (-3)^1 + 5^1 = 2 \cdot (-3) + 5 = -1$
c) $a_4 = 2 \cdot (-3)^4 + 5^4 = 2 \cdot 81 + 625 = 787$ **d)** $a_5 = 2 \cdot (-3)^5 + 5^5 = 2 \cdot (-243) + 3125 = 2639$

5. In each case we just follow the instructions.
a) $2, 5, 8, 11, 14, 17, 20, 23, 26, 29$ **b)** $1, 1, 1, 2, 2, 2, 3, 3, 3, 4$ **c)** $1, 1, 3, 3, 5, 5, 7, 7, 9, 9$
d) This requires a bit of routine calculation. For example, the fifth term is $5! - 2^5 = 120 - 32 = 88$. The first ten terms are $-1, -2, -2, 8, 88, 656, 4912, 40064, 362368, 3627776$.
e) $3, 6, 12, 24, 48, 96, 192, 384, 768, 1536$ **f)** $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$
g) For $n = 1$, the binary expansion is 1, which has one bit, so the first term of the sequence is 1. For $n = 2$, the binary expansion is 10, which has two bits, so the second term of the sequence is 2. Continuing in this way we see that the first ten terms are $1, 2, 2, 3, 3, 3, 3, 4, 4, 4$. Note that the sequence has one 1, two 2's, four 3's, eight 4's, as so on, with 2^{k-1} copies of k.
h) The English word for 1 is "one" which has three letters, so the first term is 3. This makes a good brain-teaser; give someone the sequence and ask her or him to find the pattern. The first ten terms are $3, 3, 5, 4, 4, 3, 5, 5, 4, 3$.

7. One pattern is that each term is twice the preceding term. A formula for this would be that the n^{th} term is 2^{n-1}. Another pattern is that we obtain the next term by adding increasing values to the previous term. Thus to move from the first term to the second we add 1; to move from the second to the third we add 2; then add 3, and so on. So the sequence would start out $1, 2, 4, 7, 11, 16, 22, \ldots$. We could also have trivial answers such as the rule that the first three terms are $1, 2, 4$ and all the rest are 17 (so the sequence is $1, 2, 4, 17, 17, 17, \ldots$), or that the terms simply repeat $1, 2, 4, 1, 2, 4, 1, 2, 4, \ldots$. Here is another pattern: Take n points on the unit circle, and connect each of them to all the others by line segments. The inside of the circle will be divided into a number of regions. What is the largest this number can be? Call that value a_n. If there is one point, then there are no lines and therefore just the one original region inside the circle; thus $a_1 = 1$. If $n = 2$, then the one chord divides the interior into two parts, so $a_2 = 2$. Three points give us a triangle, and that makes four regions (the inside of the triangle and the three pieces outside the triangle), so $a_3 = 4$. Careful drawing shows that the sequence starts out $1, 2, 4, 8, 16, 31$. That's right: 31, not 32.

Creative students may well find other rules or patterns with various degrees of appeal.

9. In some sense there are no right answers here. The solutions stated are the most appealing patterns that the author has found.

a) It looks as if we have one 1 and one 0, then two of each, then three of each, and so on, increasing the number of repetitions by one each time. Thus we need three more 1's (and then four 0's) to continue the sequence.

b) A pattern here is that the positive integers are listed in increasing order, with each even number repeated. Thus the next three terms are $9, 10, 10$.

c) The terms in the odd locations (first, third, fifth, etc.) are just the successive terms in the geometric sequence that starts with 1 and has ratio 2, and the terms in the even locations are all 0. The n^{th} term is 0 if n is even and is $2^{(n-1)/2}$ if n is odd. Thus the next three terms are $32, 0, 64$.

d) The first term is 3 and each successive term is twice its predecessor. This is a geometric sequence. The n^{th} term is $3 \cdot 2^{n-1}$. Thus the next three terms are $384, 768, 1536$.

e) The first term is 15 and each successive term is 7 less than its predecessor. This is an arithmetic sequence. The n^{th} term is $22 - 7n$. Thus the next three terms are $-34, -41, -48$.

f) The rule is that the first term is 3 and the n^{th} term is obtained by adding n to the $(n-1)^{\text{th}}$ term. One can actually find a quadratic expression for a sequence in which the successive differences form an arithmetic sequence; here it is $(n^2 + n + 4)/2$. The easiest way to see this is to note that the n^{th} term is $3 + 2 + 3 + 4 + 5 + 6 + \cdots + n$. Except for the initial 3 instead of a 1, the n^{th} term is the sum of the first n positive integers, which is $n(n+1)/2$ by a formula in Table 2. Therefore the n^{th} term is $(n(n+1)/2) + 2$, as claimed. We see that the next three terms are $57, 68, 80$.

g) One should play around with the sequence if nothing is apparent at first. Here we note that all the terms are even, so if we divide by 2 we obtain the sequence $1, 8, 27, 64, 125, 216, 343, \ldots$. This sequence appears in Table 1; it is the cubes. So the n^{th} term is $2n^3$. Thus the next three terms are $1024, 1458, 2000$.

h) These terms look close to the terms of the sequence whose n^{th} term is $n!$ (see Table 1). In fact, we see that the n^{th} term here is $n! + 1$. Thus the next three terms are $362881, 3628801, 39916801$.

17. The easiest way to do these sums, since the number of terms is reasonably small, is just to write out the summands explicitly. Note that the inside index (j) runs through all of its values for each value of the outside index (i).

a) $(1+1) + (1+2) + (1+3) + (2+1) + (2+2) + (2+3) = 21$

b) $(0+3+6+9) + (2+5+8+11) + (4+7+10+13) = 78$

c) $(1+1+1) + (2+2+2) + (3+3+3) = 18$

d) $(0+0+0) + (1+2+3) + (2+4+6) = 18$

23. This exercise is like Example 15. From Table 2 we know that $\sum_{k=1}^{200} k = 200 \cdot 201/2 = 20100$, and $\sum_{k=1}^{99} k = 99 \cdot 100/2 = 4950$. Therefore the desired sum is $20100 - 4950 = 15150$.

31. **a)** The negative integers are countable. Each negative integer can be paired with its absolute value to give the desired one-to-one correspondence: $1 \leftrightarrow -1$, $2 \leftrightarrow -2$, $3 \leftrightarrow -3$, etc.

b) The even integers are countable. We can list the set of even integers in the order $0, 2, -2, 4, -4, 6, -6, \ldots$, and pair them with the positive integers listed in their natural order. Thus $1 \leftrightarrow 0$, $2 \leftrightarrow 2$, $3 \leftrightarrow -2$, $4 \leftrightarrow 4$, etc. There is no need to give a formula for this correspondence—the discussion given is quite sufficient; but it is not hard to see that we are pairing the positive integer n with the even integer $f(n)$, where $f(n) = n$ if n is even and $f(n) = 1 - n$ if n is odd.

c) The proof that the set of real numbers between 0 and 1 is not countable (Example 21) can easily be modified to show that the set of real numbers between 0 and $1/2$ is not countable. We need to let the digit d_i be something like 2 if $d_{ii} \neq 2$ and 3 otherwise. The number thus constructed will be a real number between 0 and $1/2$ that is not in the list.

d) This set is countable, exactly as in part **(b)**; the only difference is that there we are looking at the multiples of 2 and here we are looking at the multiples of 7. The correspondence is given by pairing the positive integer n with $7n/2$ if n is even and $-7(n-1)/2$ if n is odd.