Homework #2, Math 266, Summer 2010

9. We need to be careful to put the lover first and the lovee second as arguments in the propositional function L.

a) $\forall x L(x, \text{Jerry})$

b) Note that the "somebody" being loved depends on the person doing the loving, so we have to put the universal quantifier first: $\forall x \exists y L(x, y)$.

c) In this case, one lovee works for all lovers, so we have to put the existential quantifier first: $\exists y \forall x L(x, y)$.

d) We could think of this as saying that there does not exist anyone who loves everybody ($\neg \exists x \forall y L(x, y)$), or we could think of it as saying that for each person, we can find a person whom he or she does not love ($\forall x \exists y \neg L(x, y)$. These two expressions are logically equivalent.

e) $\exists x \neg L(\text{Lydia}, x)$

f) We are asserting the existence of an individual such that everybody fails to love that person: $\exists x \forall y \neg L(y, x)$.

g) In Exercise 52 of Section 1.3, we saw that there is a notation for the existence of a unique object satisfying a certain condition. Employing that device, we could write this as $\exists! x \forall y L(y, x)$. In Exercise 52 of the present section we will discover a way to avoid this notation in general. What we have to say is that the x asserted here exists, and that every z satisfying this condition (of being loved by everybody) must equal x. Thus we obtain $\exists x (\forall y L(y, x) \land \forall z ((\forall w L(w, z)) \rightarrow z = x))$. Note that we could have used y as the bound variable where we used w; since the scope of the first use of y had ended before we came to this point in the formula, reusing y as the bound variable would cause no ambiguity.

h) We want to assert the existence of two distinct people, whom we will call x and y, whom Lynn loves, as well as make the statement that everyone whom Lynn loves must be either x or y: $\exists x \exists y (x \neq y \land L(\text{Lynn}, x) \land L(\text{Lynn}, y) \land \forall z (L(\text{Lynn}, z) \rightarrow (z = x \lor z = y)))$.

i) $\forall x L(x, x)$ (Note that nothing in our earlier answers ruled out the possibility that variables or constants with different names might be equal to each other. For example, in part **(a)**, x could equal Jerry, so that statement includes as a special case the assertion that Jerry loves himself. Similarly, in part **(h)**, the two people whom Lynn loves either could be two people other than Lynn (in which case we know that Lynn does not love herself), or could be Lynn herself and one other person.)

j) This is asserting that the one and only one person who is loved by the person being discussed is in fact that person: $\exists x \forall y (L(x, y) \leftrightarrow x = y)$.

27. Recall that the integers include the positive and negative integers and 0.

a) The import of this statement is that no matter how large n might be, we can always find an integer m bigger than n^2. This is certainly true; for example, we could always take $m = n^2 + 1$.

b) This statement is asserting that there is an n that is smaller than the square of *every* integer; note that n is not allowed to depend on m, since the existential quantifier comes first. This statement is true, since we could take, for instance, $n = -3$, and then n would be less than every square, since squares are always greater than or equal to 0.

c) Note the order of quantifiers: m here is allowed to depend on n. Since we can take $m = -n$, this statement is true (additive inverses exist for the integers).

d) Here one n must work for all m. Clearly $n = 1$ does the trick, so the statement is true.

e) The statement is that the equation $n^2 + m^2 = 5$ has a solution over the integers. This is true; in fact there are eight solutions, namely $n = \pm 1$, $m = \pm 2$, and vice versa.

f) The statement is that the equation $n^2 + m^2 = 6$ has a solution over the integers. There are only a small finite number of cases to try, since if $|m|$ or $|n|$ were bigger than 2 then the left-hand side would be bigger than 6. A few minutes reflection shows that in fact there is no solution, so the existential statement is false.

g) The statement is that the system of equations $\{n + m = 4, n - m = 1\}$ has a solution over the integers. By algebra we see that there is a unique solution to this system, namely $n = 2\frac{1}{2}$, $m = 1\frac{1}{2}$. Since there do not exist *integers* that make the equations true, the statement is false.

h) The statement is that the system of equations $\{n + m = 4, n - m = 2\}$ has a solution over the integers. By algebra we see that there is indeed an integral solution to this system, namely $n = 3$, $m = 1$. Therefore the statement is true.

i) This statement says that the average of two integers is always an integer. If we take $m = 1$ and $n = 2$, for example, then the only p for which $p = (m + n)/2$ is $p = 1\frac{1}{2}$, which is not an integer. Therefore the statement is false.

45. This statement says that every number has a multiplicative inverse.

a) In the universe of nonzero real numbers, this is certainly true. In each case we let $y = 1/x$.

b) Integers usually don't have inverses that are integers. If we let $x = 3$, then no integer y satisfies $xy = 1$. Thus in this setting, the statement is false.

c) As in part **(a)** this is true, since $1/x$ is positive when x is positive.

9. a) Because it was sunny on Tuesday, we assume that it did not rain or snow on Tuesday (otherwise we cannot do anything with this problem). If we use modus tollens on the universal instantiation of the given conditional statement applied to Tuesday, then we conclude that I did not take Tuesday off. If we now apply disjunctive syllogism to the disjunction in light of this conclusion, we see that I took Thursday off. Now use modus ponens on the universal instantiation of the given conditional statement applied to Thursday; we conclude that it rained or snowed on Thursday. One more application of disjunctive syllogism tells us that it rained on Thursday.

b) Using modus tollens we conclude two things—that I did not eat spicy food and that it did not thunder. Therefore by the conjunction rule of inference (Table 1), we conclude "I did not eat spicy food and it did not thunder."

c) By disjunctive syllogism from the first two hypotheses we conclude that I am clever. The third hypothesis gives us no useful information.

d) We can apply universal instantiation to the conditional statement and conclude that if Ralph (respectively, Ann) is a CS major, then he (she) has a PC. Now modus tollens tells us that Ralph is not a CS major. There are no conclusions to be drawn about Ann.

e) The first two conditional statements can be phrased as "If x is good for corporations, then x is good for the U.S." and "If x is good for the U.S., then x is good for you." If we now apply universal instantiation with x being "for you to buy lots of stuff," then we can conclude using modus ponens twice that for you to buy lots of stuff is good for the U.S. and is good for you.

f) The given conditional statement is "For all x, if x is a rodent, then x gnaws its food." We can form the universal instantiation of this with x being a mouse, a rabbit, and a bat. Then modus ponens allows us to conclude that mice gnaw their food; and modus tollens allows us to conclude that rabbits are not rodents. We can conclude nothing about bats.

15. a) This is correct, using universal instantiation and modus ponens.

b) This is invalid. After applying universal instantiation, it contains the fallacy of affirming the conclusion.

c) This is invalid. After applying universal instantiation, it contains the fallacy of denying the hypothesis.

d) This is valid by universal instantiation and modus tollens.

19. a) This is the fallacy of affirming the conclusion, since it has the form "$p \to q$ and q implies p."

b) This reasoning is valid; it is modus tollens.

c) This is the fallacy of denying the hypothesis, since it has the form "$p \to q$ and $\neg p$ implies $\neg q$."

24. Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

27. We can set this up in two-column format.

Step	**Reason**
1. $\forall x(P(x) \wedge R(x))$	Premise
2. $P(a) \wedge R(a)$	Universal instantiation using (1)
3. $P(a)$	Simplification using (2)
4. $\forall x(P(x) \to (Q(x) \wedge S(x)))$	Premise
5. $Q(a) \wedge S(a)$	Universal modus ponens using (3) and (4)
6. $S(a)$	Simplification using (5)
7. $R(a)$	Simplification using (2)
8. $R(a) \wedge S(a)$	Conjunction using (7) and (6)
9. $\forall x(R(x) \wedge S(x))$	Universal generalization using (5)

6. An odd number is one of the form $2n+1$, where n is an integer. We are given two odd numbers, say $2a+1$ and $2b+1$. Their product is $(2a+1)(2b+1) = 4ab+2a+2b+1 = 2(2ab+a+b)+1$. This last expression shows that the product is odd, since it is of the form $2n+1$, with $n = 2ab+a+b$.

15. We will prove the contrapositive (that if it is not true that $x \geq 1$ or $y \geq 1$, then it is not true that $x+y \geq 2$), using a direct argument. Assume that it is not true that $x \geq 1$ or $y \geq 1$. Then (by De Morgan's law) $x < 1$ and $y < 1$. Adding these two inequalities, we obtain $x+y < 2$. This is the negation of $x+y \geq 2$, and our proof is complete.

17. **a)** We must prove the contrapositive: If n is odd, then n^3+5 is even. Assume that n is odd. Then we can write $n = 2k+1$ for some integer k. Then $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$. Thus n^3+5 is two times some integer, so it is even.

b) Suppose that n^3+5 is odd and that n is odd. Since n is odd, and the product of odd numbers is odd, in two steps we see that n^3 is odd. But then subtracting we conclude that 5, being the difference of the two odd numbers n^3+5 and n^3, is even. This is not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

26. We need to prove two things, since this is an "if and only if" statement. First let us prove directly that if n is even then $7n+4$ is even. Since n is even, it can be written as $2k$ for some integer k. Then $7n+4 = 14k+4 = 2(7k+2)$. This is 2 times an integer, so it is even, as desired. Next we give a proof by contraposition that if $7n+4$ is even then n is even. So suppose that n is not even, i.e., that n is odd. Then n can be written as $2k+1$ for some integer k. Thus $7n+4 = 14k+11 = 2(7k+5)+1$. This is 1 more than 2 times an integer, so it is odd. That completes the proof by contraposition.

29. This proposition is true. We give a proof by contradiction. Suppose that m is neither 1 nor -1. Then mn has a factor (namely $|m|$) larger than 1. On the other hand, $mn = 1$, and 1 clearly has no such factor. Therefore we conclude that $m = 1$ or $m = -1$. It is then immediate that $n = 1$ in the first case and $n = -1$ in the second case, since $mn = 1$ implies that $n = 1/m$.

33. It is easiest to give proofs by contraposition of $(i) \rightarrow (ii)$, $(ii) \rightarrow (i)$, $(i) \rightarrow (iii)$, and $(iii) \rightarrow (i)$. For the first of these, suppose that $3x+2$ is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x = ((p/q)-2)/3 = (p-2q)/(3q)$, where $3q \neq 0$. This shows that x is rational. For the second conditional statement, suppose that x is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $3x+2 = (3p+2q)/q$, where $q \neq 0$. This shows that $3x+2$ is rational. For the third conditional statement, suppose that $x/2$ is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x = 2p/q$, where $q \neq 0$. This shows that x is rational. And for the fourth conditional statement, suppose that x is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x/2 = p/(2q)$, where $2q \neq 0$. This shows that $x/2$ is rational.