4/9/2024

Separable Differential Equations (4.3)
Logistic Equations/Population Models (4.4)

Separable differential equations are equations that we can algebraically put all the y-variables on one side of the equation, and all the independent variables (x or t) on the other side of the equation. (in some ways, we can think of this as undoing a chain rule or undoing implicit differentiation).

In some cases, these problems are autonomous (no independent variable explicitly in the equation). In some cases, these problems are nonlinear.

$$\frac{dP}{dt} = kP$$

The population is increasing at a rate proportional to the existing population (k is positive).

P is the function variable, so we want to do algebra to get dP and P on the same side of the equation, and get t and any other components to the other side of the equation.

Multiply both sides of the equation by dt, and divide both sides by P

$$\frac{dP}{P} = kdt$$

$$\frac{1}{P}dP = kdt$$

Integrate both sides with respect to the variable on each side.

$$\int \frac{1}{P}dP = \int k\ dt$$

$$\ln|P| = kt + C$$

$$e^{\ln|P|} = e^{kt+C}$$
$$P = e^{kt}e^{C}$$
$$e^{C} = P_0$$

$$P(t) = P_0e^{kt}$$

Standard exponential growth function.

Example.

$$\frac{dy}{dx} = 3x^2y + 2x^2 - 12y - 8$$

$$\frac{dy}{dx} = x^2(3y+2) - 4(3y+2)$$

$$\frac{dy}{dx} = (x^2 - 4)(3y + 2)$$

$$\frac{dy}{3y + 2} = (x^2 - 4)dx$$

$$\int \frac{dy}{3y + 2} = \int x^2 - 4\ dx$$

$$u = 3y + 2, du = 3dy \to \frac{1}{3}du = dy$$

$$\int \frac{1}{3} \times \left(\frac{1}{u}\right) du = \int x^2 - 4\ dx$$

$$\frac{1}{3}\ln|u| = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|3y + 2| = \frac{1}{3}x^3 - 4x + C$$

Not every problem must be solved for the function variable. It is okay in some cases to leave it in implicit form.

$$\ln|3y + 2| = x^3 - 12x + C$$
$$3y + 2 = e^{x^3 - 12x + C}$$
$$3y + 2 = Y_0 e^{x^3 - 12x}$$
$$3y = Y_0 e^{x^3 - 12x} - 2$$
$$y(x) = A_0 e^{x^3 - 12x} - \frac{2}{3}$$

Example.

$$\frac{dy}{dx} = (2x + 3)(y^2 - 4), y(0) = -3$$

(when an initial value like y(0) is present, this is called an initial value problem, and we use the extra information to find the value of the constant).

$$\frac{dy}{dx} = (2x + 3)(y^2 - 4)$$

$$\frac{dy}{y^2 - 4} = 2x + 3\ dx$$

$$\int \frac{dy}{y^2 - 4} = \int 2x + 3\ dx$$

$$\int \frac{dy}{(y-2)(y+2)} = x^2 + 3x + C$$

$$\frac{A}{y-2} + \frac{B}{y+2} = \frac{\big(A(y+2) + B(y-2)\big)}{(y-2)(y+2)}$$

$$Ay + 2A + By - 2B = 1$$

$$A + B = 0$$
$$2A - 2B = 1$$
$$A - B = \frac{1}{2}$$
$$A + B = 0$$
$$2A = \frac{1}{2} \to A = \frac{1}{4}$$

$$B = -\frac{1}{4}$$

$$\int \frac{dy}{(y-2)(y+2)} = \int \frac{\frac{1}{4}}{y-2} + \frac{\left(-\frac{1}{4}\right)}{y+2} dy = \frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x^2 + 3x + C$$

$$\ln|y-2| - \ln|y+2| = 4x^2 + 12x + C$$
$$\ln\left|\frac{y-2}{y+2}\right| = 4x^2 + 12x + C$$

$$\frac{y-2}{y+2} = Ae^{4x^2+12x}$$

$$y - 2 = Aye^{4x^2+12x} + 2Ae^{4x^2+12x}$$

$$y - Aye^{4x^2+12x} = 2 + 2Ae^{4x^2+12x}$$

$$y\left(1 - Ae^{4x^2+12x}\right) = 2 + 2Ae^{4x^2+12x}$$

$$y(x) = \frac{2 + 2Ae^{4x^2+12x}}{1 - Ae^{4x^2+12x}}$$

$$y(0) = -3$$

$$\frac{y-2}{y+2} = Ae^{4x^2+12x}$$

$$-\frac{5}{-1} = \frac{-3-2}{-3+2} = Ae^{0+0} = A$$

$$A = 5$$

$$\frac{y-2}{y+2} = 5e^{4x^2+12x}$$

$$y(x) = \frac{2+10e^{4x^2+12x}}{1-5e^{4x^2+12x}}$$

Example.
A tank that contains a 100L of brine solution that initially has 4kg of salt in it. Pumping fluid into the tank (t=0), at a rate of 2L/min, containing 0.5kg/L of salt. The water is flowing out of the tank (after being mixed well) at a rate of 2L/min. Find an equation that tells us how much sale is in the tank at any time t.

The rate of change of salt = rate in minus the rate out

$$\frac{dS}{dt} = \frac{2L}{min} \times 0.5\frac{kg}{L} - \frac{2L}{min} \times \frac{S}{100L}$$

$$\frac{dS}{dt} = 1 - \frac{S}{50}, A(0) = 4$$

$$\frac{dS}{dt} = \frac{1}{50}(50 - S) = -\frac{1}{50}(S - 50)$$

$$\frac{dS}{S-50} = -\frac{1}{50}dt$$

$$\int \frac{dS}{S-50} = \int -\frac{1}{50}dt$$

$$\ln|S - 50| = -\frac{1}{50}t + C$$

$$S - 50 = Ae^{-\frac{1}{50}t}$$

$$S(t) = 50 + Ae^{-\frac{1}{50}t}$$

$$S(0) = 4 = 50 + Ae^0 \rightarrow A = -46$$

$$S(t) = 50 - 46e^{-\frac{1}{50}t}$$

What is the amount of salt in the tank at $t = 10\ minutes$?

$$S(10) = 50 - 46 = 12.338 \ldots kg$$

What is the equilibrium amount of salt in the tank? What is the maximum level of salt that can be achieved through this process? (Where is this going over time?)

The limiting level of salt in the tank is 50kg.

Logistic equations are similar to a previous example. They are autonomous, and typically have (factorable) polynomials in their differential equations.

$$\frac{dy}{dt} = ky(y - 10)$$

If we solve these, we use partial fractions, and if there are more than two factors, they are almost impossible to solve explicitly for y.

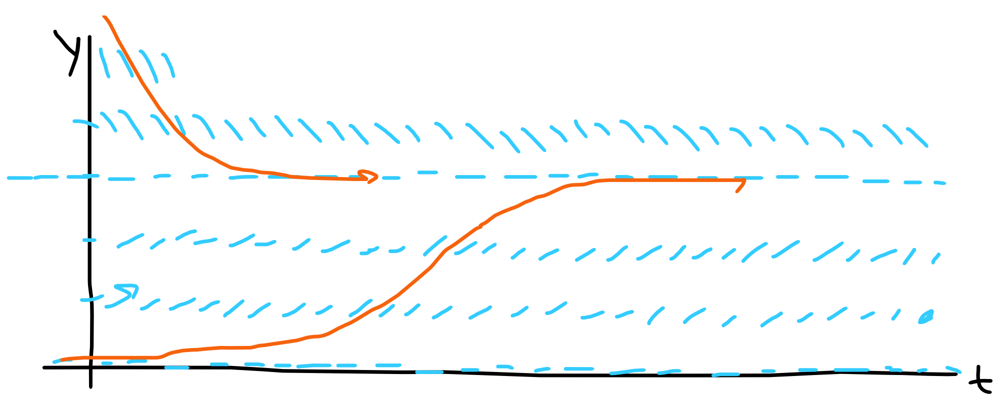
Direction fields for autonomous equations are much easier to plot because they only differ by y and not by x (or t). They change vertically, but not horizontally.

$$\frac{dy}{dt} = -y(y - 3)$$

Population models don't generally graph values below zero, because negative population doesn't usually mean anything.

What are important points? Equilibria are where the differential equation is equal to zero.

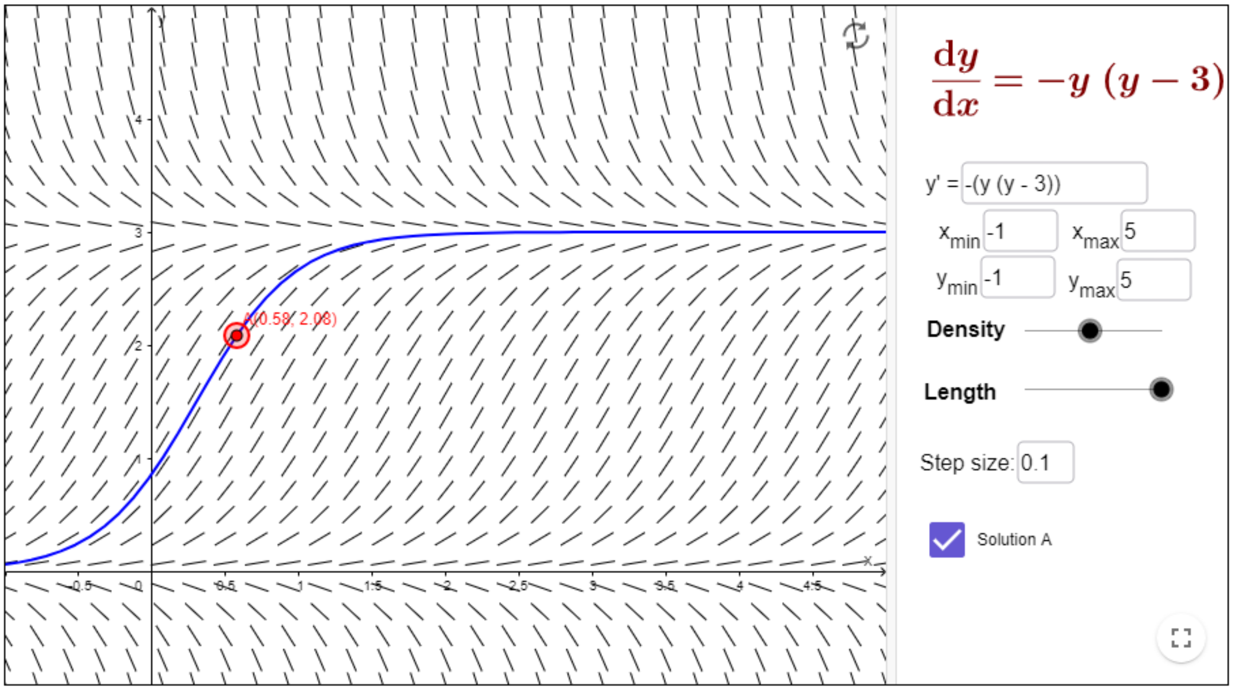
$$0 = -y(y - 3)$$
$$y = 0, y = 3$$



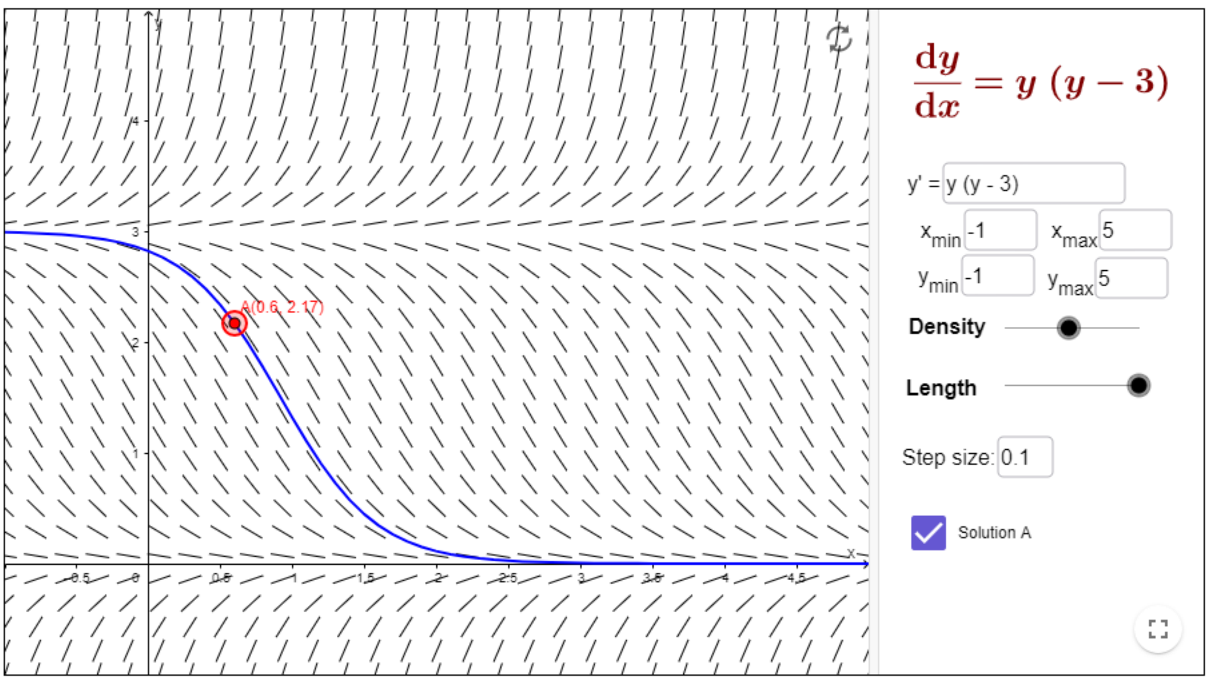
Equilibria can have different behaviors (for population models we usually only characterize equilibria that are y>0)

1) The equilibrium is an attractor (slopes point toward the equilibrium), carrying capacity, stable
2) The equilibrium is a repeller (slopes point away from the equilibrium), threshold, unstable
3) The equilibrium could be semi-stable when slopes on one side point away from the equilibrium and point toward it on the other.

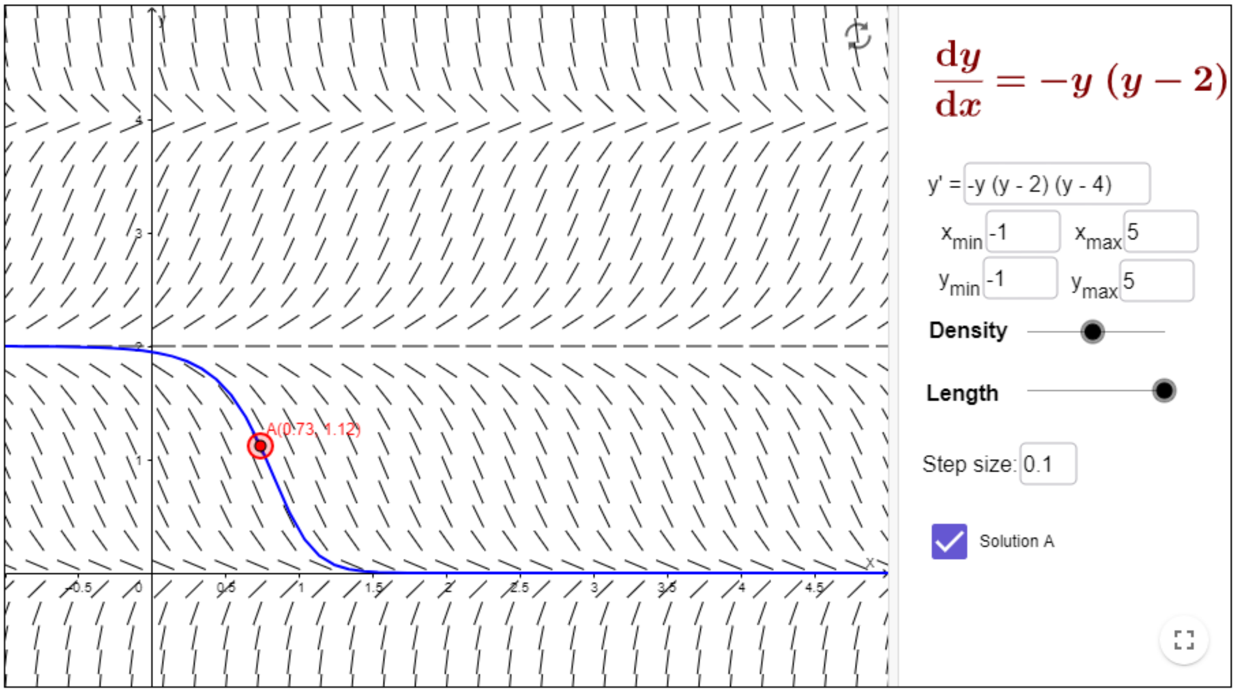
https://www.geogebra.org/m/Pd4Hn4BR

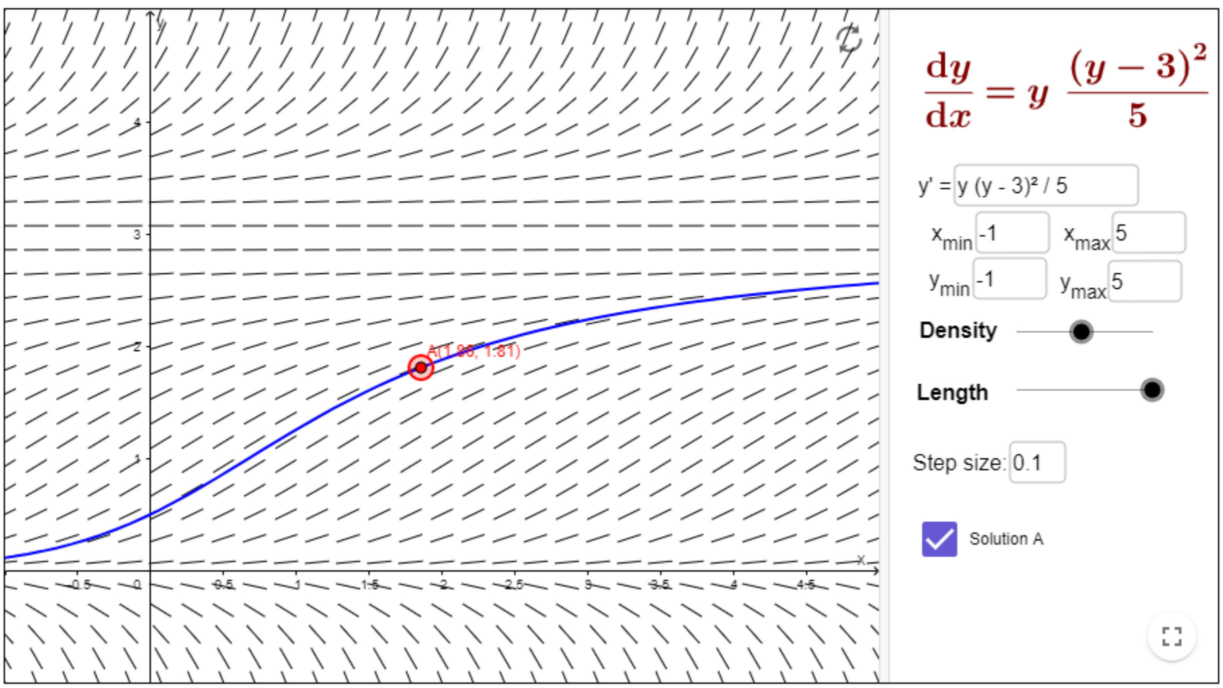


A(0.58, 2.08)
$$\frac{dy}{dx} = -y\,(y-3)$$
y' = -(y (y - 3))
x_{min} -1
x_{max} 5
y_{min} -1
y_{max} 5
Density
Length
Step size: 0.1
Solution A



A(0.6, 2.17)
$$\frac{dy}{dx} = y\,(y-3)$$
y' = y (y - 3)
x_{min} -1
x_{max} 5
y_{min} -1
y_{max} 5
Density
Length
Step size: 0.1
Solution A


$\frac{dy}{dx} = -y\ (y-2)$
y' = -y (y - 2) (y - 4)
x_{min} -1 x_{max} 5
y_{min} -1 y_{max} 5
Density
Length
Step size: 0.1
Solution A
A(0.73, 1.12)


$\frac{dy}{dx} = y\ \frac{(y-3)^2}{5}$
y' = y (y - 3)² / 5
x_{min} -1 x_{max} 5
y_{min} -1 y_{max} 5
Density
Length
Step size: 0.1
Solution A