4/4/2024

Introduction to Differential Equations (4.1)
Direction Fields and Numerical Methods (4.2)

Differential Equation is an equation that contains a variable (function) and its derivatives. It may or may not include the independent variable of the function.

$$\frac{dy}{dt} = ky$$

The rate of change of y is linearly proportional to the value of the function at that point.

This is an autonomous equation because it does not contain the independent variable: $y' = ky$

$$y'' + y' + y = \cos(t)$$

We can classify our differential equations in several dimensions:

Ordinary vs. partial differential equations

A partial derivative is a derivative of a function for a particular variable, but the function contains more than one independent variable.
Notation for a partial derivative uses ∂ instead of d

The ordinary derivative for y with respect to t is $\frac{dy}{dt}$
The partial derivative for y with respect to t is $\frac{\partial y}{\partial t}$

Ordinary derivative can be noted with the prime symbol: y'
Partial derivative can be noted with subscripts to indicated the derivative variable: z_y, z_x

$$\frac{\partial z}{\partial x} = z_x, \frac{\partial z}{\partial y} = z_y$$

Example.

$$y'' + t^2 y' = \cot(y)$$
$$\frac{dy}{dt} = y^7 + \sqrt{t}$$

Both of these are ordinary differential equations.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$u_{xx} + u_{yy} = u_{xy}$$

These are both partial differential equations.

Order of the differential equation: this is determined the highest degree of the derivative in the equation.

A first-order problem contains only the first derivative.
A second-order problem contains up to the second derivative

Example.

$$y' + \sin(t)\, y = e^t$$
$$\left(\frac{dy}{dt}\right)^2 = \tan(t) + 11$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = e^x$$
$$u_x + u_y + u_z = 0$$

Second order:

$$y'' + 3y' + y = t^2$$
$$u_{xx} - u_y = 0$$
$$u_{xy} = \cos(u)$$

Third order:

$$y''' + (y')^3 = 0$$
$$\frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial u}{\partial x}$$

Linearity: is the differential linear or nonlinear?
The independent variable can do whatever for this question, what matters is what the function and its derivatives.
As long as the function (or its derivatives) are not inside other nonlinear functions, being raised to powers, or multiplying each other, it's considered linear.

$$y'' = ky$$

But:

$$y'' = \sin(t)\, y$$
$$y' = \tan(t) + 11$$

Is also linear.

What is not linear?

$$\left(\frac{dy}{dt}\right)^2 = \tan(t) + 11$$
$$y'' = \sin\,(y)$$
$$(y')^3 = y'''$$
$$\sin(y) = y'' + y'$$
$$u_{xx} + u_{yy} = u_x u_y + e^u + \sin\,(u)$$
$$y''(y) = \ln(t)$$

Given a differential equation, determine whether it is i) linear or non-linear, ii) partial or ordinary, iii) the order.

$$y'' + \left(\frac{1}{t}\right)y' + t^2 y = e^t$$

$$u_x u_y + xy = u^2$$

Verifying solutions to a differential equation.

Take the proposed solution, plug it into the equation and show that it creates an identity equation.

Verify that the solution to the differential equation $y'' - 3y' + 2y = 24e^{-2x}$ is $y = 3e^x - 4e^{2x} + 2e^{-2x}$

$$y' = 3e^x - 8e^{2x} - 4e^{-2x}$$
$$y'' = 3e^x - 16e^{2x} + 8e^{-2x}$$

$$y'' - 3y' + 2y = 3e^x - 16e^{2x} + 8e^{-2x} - 3(3e^x - 8e^{2x} - 4e^{-2x}) + 2(3e^x - 4e^{2x} + 2e^{-2x}) =$$
$$3e^x - 16e^{2x} + 8e^{-2x} - 9e^x + 24e^{2x} + 12e^{-2x} + 6e^x - 8e^{2x} + 4e^{-2x} =$$
$$24e^{-2x}$$

Checks out.

May also need to verify an initial condition, for example some value of the function when t (or x) is a particular value such as 0. $y(0) = 1$.

Some differential equations can be solved just by integration, if you have $y' = f(t)$. Then the solution to the equation is just the antiderivative of $f(t)$.

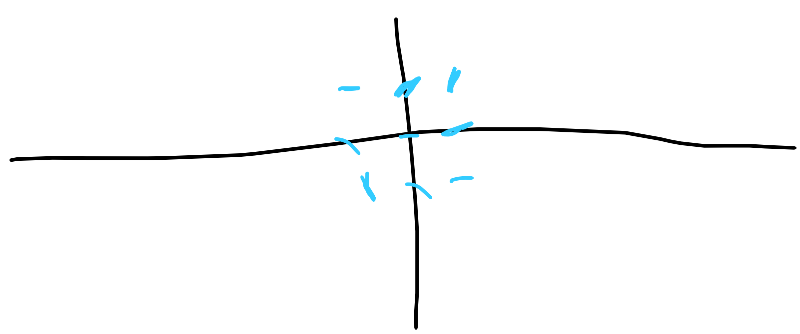
$$y = \int f(t)dt + C$$

Direction field or slope field:
A graph of the little slopes in the plane defined by a differential equation.
Graphically representing how the differential equation is telling a particular to behave (where to go next) if it has a particular value at the start.

$$y' = x + y$$

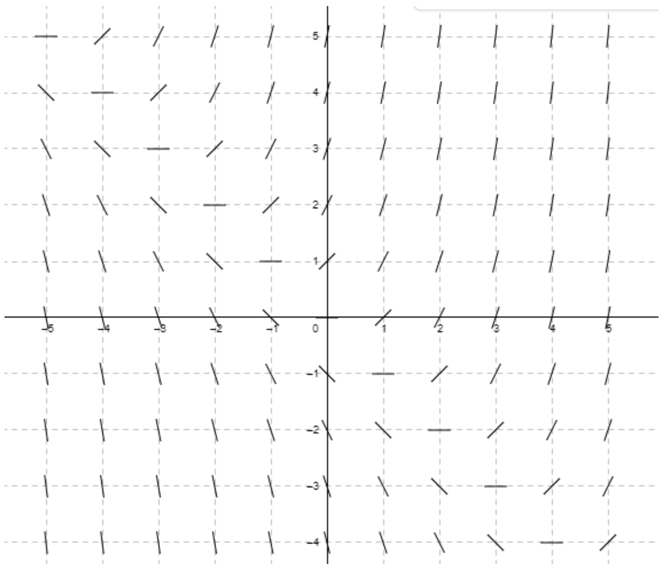
For any pair of x and y, the equation tells me the slope of the graph at that point.

x	y	y'
0	0	0
1	0	1
0	1	1
0	-1	-1
-1	0	-1



1	1	2
1	-1	0

https://www.geogebra.org/m/W7dAdgqc



The differential chapter is not on the MyOpenMath homework. I encourage everyone to do homework #9 (the written one).

Euler's Method
Is a numerical approximation method for estimating the behavior of a solution to a differential equation without solving for the exact solution.

A series of linear approximations, follow the tangent line (whose slope is predicted by the differential equation) for a fixed time period, and then a new direction is estimated from the differential equation at that new point.

$$y' = f(t, y)$$
$$y_{n+1} = f(t_n, y_n)\Delta t + y_n$$

$$y' = t + y$$

Estimate the value of $y(3)$, if we know that $y(0) = 1$, estimate in 3 steps.

$$\Delta t = \frac{3-0}{3} = 1$$

$$t_0 = 0, y_0 = 1$$
$$m_0 = f(t_0, y_0) = 0 + 1 = 1$$
$$y_1 = (1)(1) + 1 = 2$$

$$t_1 = 1, y_1 = 2$$
$$m_1 = f(t_1, y_1) = 1 + 2 = 3$$
$$y_2 = (3)(1) + 2 = 5$$

$$t_2 = 2, y_2 = 5$$
$$m_2 = f(t_2, y_2) = 2 + 5 = 7$$
$$y_3 = (7)(1) + 5 = 12$$

$$t_3 = 3, y_3 = 12$$

Estimate $y(3) \approx 12$