4/23/2024

Calculus in Polar Coordinates (7.4)

Finding the slope of the tangent line.
(equation of the tangent line will need a point on the curve in rectangular coordinates. You will need to switch the point on the polar curve to rectangular coordinates. We rarely ask for the equation of tangent line because of these extra complications.)

Recall:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Parametrize a circle:

$$x = r\cos t\,, y = r\sin t$$

(t = θ)
In polar form: a circle has a constant radius and is just given by $r = a$

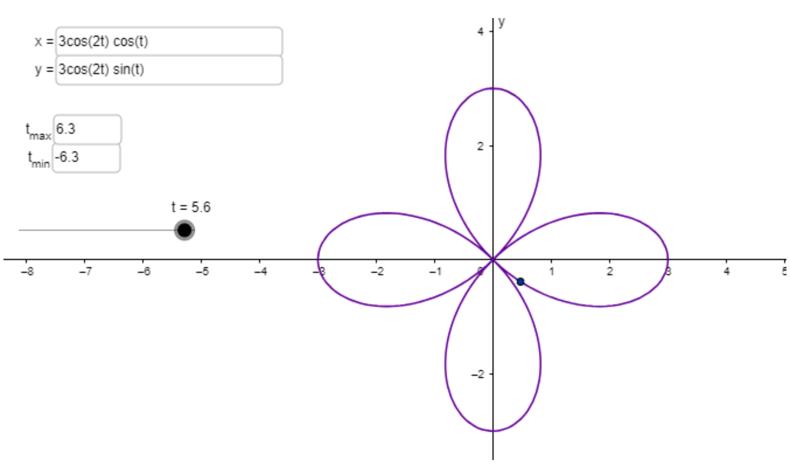
$$x^2 + y^2 = 9$$
$$(r\cos t)^2 + (r\sin t)^2 = 9$$
$$r^2\cos^2 t + r^2\sin^2 t = 9$$
$$r^2(\cos^2 t + \sin^2 t) = 9$$
$$r^2 = 9$$
$$r = 3$$

$$x = 3\cos t\,, y = 3\sin t$$

We can parametrize any polar curve in terms of t (θ). That is, replace the "r" in the circle parametrization with $r(t) = r(\theta)$.

Suppose we want to write the equation $r = 3\cos(2\theta)$ in parametric form.

$$x = 3\cos(2t)\cos(t)\,, y = 3\cos(2t)\sin t$$



$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$x(t) = r(t)\cos(t)\,, y(t) = r(t)\sin\,(t)$$

$$\frac{dx}{dt} = r'(t)\cos t - r(t)\sin t$$

$$\frac{dy}{dt} = r'(t)\sin(t) + r(t)\cos(t)$$

$$\frac{dy}{dx} = \frac{r'(t)\sin(t) + r(t)\cos(t)}{r'(t)\cos t - r(t)\sin t} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

$$r(\theta) = 3\cos\,(2\theta)$$

Evaluate the slope of the tangent line at $\frac{\pi}{3}$

$$r'(\theta) = -6\sin(2\theta)$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$r\left(\frac{\pi}{3}\right) = 3\cos\left(\frac{2\pi}{3}\right) = -\frac{3}{2}, r'\left(\frac{\pi}{3}\right) = -6\sin\left(\frac{2\pi}{3}\right) = -3\sqrt{3}$$

$$\frac{dy}{dx} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos\theta - r(\theta)\sin\theta} = \frac{-3\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) - \frac{3}{2}\left(\frac{1}{2}\right)}{-3\sqrt{3}\left(\frac{1}{2}\right) + \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)} = \frac{-\frac{9}{2} - \frac{3}{4}}{-\frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{4}} = \frac{-\frac{21}{4}}{-\frac{3\sqrt{3}}{4}} = \frac{7}{\sqrt{3}}$$

You do not need to rationalize.

$$s = \int_a^b \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}d\theta$$

Find the length of arc of the circle $r = 2\sin\theta$ on the interval $[0, \pi]$.

$$s = \int_0^\pi \sqrt{[2\sin\theta]^2 + [2\cos\theta]^2}d\theta = \int_0^\pi \sqrt{4\sin^2\theta + 4\cos^2\theta}\,d\theta = \int_0^\pi \sqrt{4(\sin^2\theta + \cos^2\theta)}d\theta =$$

$$\int_0^\pi 2d\theta = 2\theta|_0^\pi = 2\pi$$

This agrees with the geometry, with radius=1 (diameter=2), and $C = 2\pi r$. $C = 2\pi$.

Example.
Find the length of arc of the polar curve $r = 2 + 2\cos\theta$

$$s = \int_0^{2\pi} \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta =$$

$$\int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4}d\theta = \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = \int_0^{2\pi} 2\sqrt{2 + 2\cos\theta}\, d\theta$$

$$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$
$$\cos(\theta) + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$$
$$2\cos(\theta) + 2 = 4\cos^2\left(\frac{\theta}{2}\right)$$

$$\int_0^{2\pi} 2\sqrt{2 + 2\cos\theta}\, d\theta = \int_0^{2\pi} 2\sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = \int_0^{2\pi} 4\left|\cos\left(\frac{\theta}{2}\right)\right| d\theta = \int_0^{\pi} 8\cos\left(\frac{\theta}{2}\right) d\theta =$$

$$8\left[\sin\left(\frac{\theta}{2}\right)(2)\right]_0^{\pi} = 16(1-0) = 16$$

Area inside a polar curve.

$$A = \frac{1}{2}\int_a^b [r(\theta)]^2 d\theta$$

Find the area of one petal of the rose $r = 3\cos(2\theta)$

Set the equation equal to zero to find the bounds on the petal.

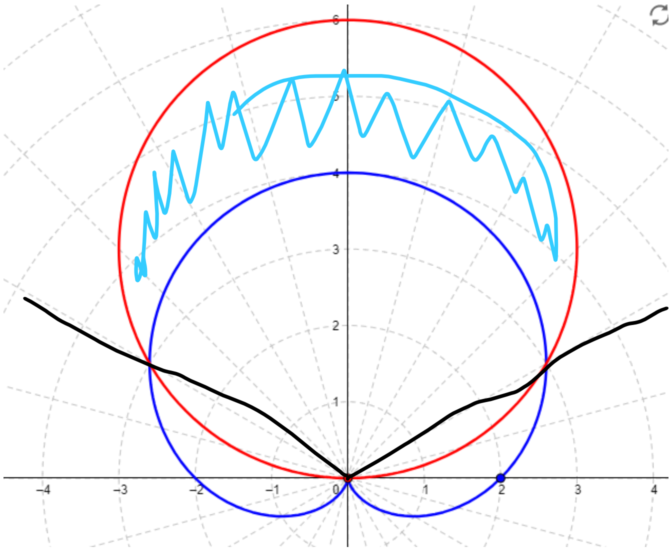
$$3\cos(2\theta) = 0$$
$$\cos(2\theta) = 0$$
$$2\theta = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, etc.$$
$$\theta = \frac{\pi}{4}, -\frac{\pi}{4}, etc.$$

$$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (3\cos 2\theta)^2 d\theta = \left(\frac{1}{2}\right)(2)\int_0^{\frac{\pi}{4}} 9\cos^2 2\theta\, d\theta = 9\int_0^{\frac{\pi}{4}} \left(\frac{1}{2}\right)(1 + \cos(4\theta))d\theta =$$

$$\frac{9}{2}\int_0^{\frac{\pi}{4}} 1 + \cos(4\theta)\, d\theta = \frac{9}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{4}} = \frac{9}{2}\left[\frac{\pi}{4}\right] = \frac{9\pi}{8}$$

Find the area between 2 polar curves.

Find the area outside the curve $r = 2 + 2\sin\theta$, and inside $r = 6\sin\theta$.



$$2 + 2\sin\theta = 6\sin\theta$$
$$2 = 4\sin\theta$$
$$\frac{1}{2} = \sin\theta$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_a^b [r_{outer}(\theta)]^2 d\theta - \frac{1}{2}\int_a^b [r_{inner}(\theta)]^2 d\theta = \frac{1}{2}\int_a^b [r_{outer}(\theta)]^2 - [r_{inner}(\theta)]^2 d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (6\,sin\,\theta)^2 - (2 + 2\sin\theta)^2 d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (6\,sin\,\theta)^2 - (2 + 2\sin\theta)^2 d\theta =$$

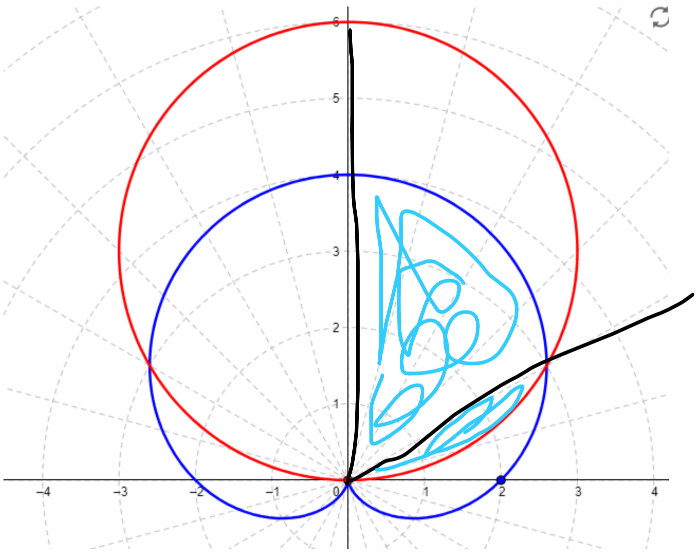
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 36\sin^2\theta - (4 + 8\sin\theta + 4\sin^2\theta)d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 32\sin^2\theta - 4 - 8\sin\theta\, d\theta =$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 32\left(\frac{1}{2}\right)(1 - \cos 2\theta) - 4 - 8\sin\theta\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 16 - 16\cos 2\theta - 4 - 8\sin\theta\, d\theta =$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 12 - 16\cos 2\theta - 8\sin\theta\, d\theta = 12\theta - 8\sin 2\theta + 8\cos\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$$

$$12\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - 8\left(0 - \frac{\sqrt{3}}{2}\right) + 8\left(0 - \frac{\sqrt{3}}{2}\right) = 12\left(\frac{\pi}{3}\right) + 4\sqrt{3} - 4\sqrt{3} = 4\pi$$

Finding the area inside both $r = 2 + 2\sin\theta$ and $r = 6\sin\theta$



$$A = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 + 2\sin\theta)^2 d\theta + \int_{0}^{\frac{\pi}{6}} (6\sin^2\theta) d\theta$$

(1/2 goes away because we are doubling it for symmetry)

Next time: conic sections in polar coordinates
Eccentricity as a means of identifying the conic section