4/16/2024

Calculus of Parametric Equations (and Vectors)

Derivatives of parametric equations (and vectors)

Example:
Consider the set of parametric equations given by

$$x(t) = t^2 - 1, y(t) = t^3 + 1$$
$$\vec{r}(t) = \langle t^2 - 1, t^3 + 1 \rangle$$
$$x'(t) = 2t, y'(t) = 3t^2$$
$$\vec{r}'(t) = \langle 2t, 3t^2 \rangle$$

Find the equation of the tangent to the graph when $t = 2$

$$slope = \frac{\Delta y}{\Delta x} = \frac{y'(t)}{x'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

Chain rule:

$$\frac{d[y(x)]}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3}{2}(2) = 3$$

Points on the original curve:

$$x(2) = (2)^2 - 1 = 3$$
$$y(2) = (2)^3 + 1 = 9$$
$$y - y_1 = m(x - x_1)$$
$$y - 9 = 3(x - 3)$$
$$y - 9 = 3x - 9$$
$$y = 3x$$

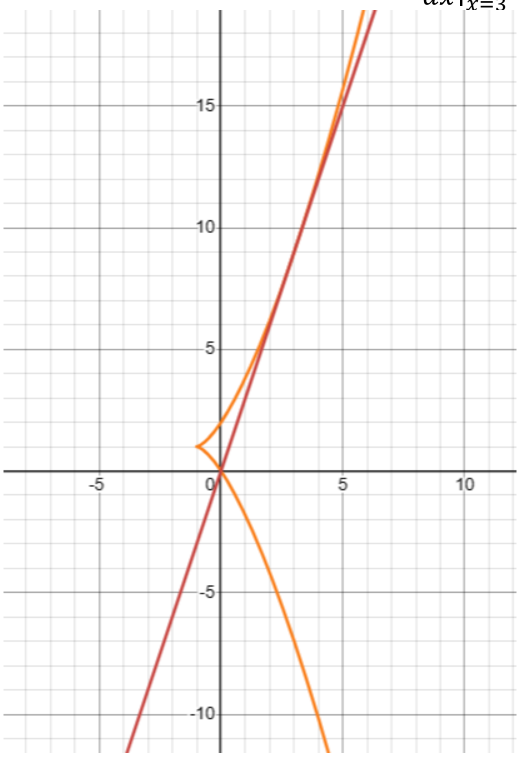
$$x(t) = t^2 - 1, y(t) = t^3 + 1$$

$$x = t^2 - 1$$
$$x + 1 = t^2$$
$$\sqrt{x + 1} = t$$

$$y = (x + 1)^{\frac{3}{2}} + 1$$

$$\frac{dy}{dx} = \left(\frac{3}{2}\right)(x+1)^{\frac{1}{2}}$$

$$\left.\frac{dy}{dx}\right|_{x=3} = \left(\frac{3}{2}\right)(4)^{\frac{1}{2}} = \left(\frac{3}{2}\right)(2) = 3$$



What about the second derivative?

$$\frac{d}{dx}\left[\frac{dy}{dx}(t)\right] = \frac{\left(\frac{d}{dt}\left[\frac{dy}{dx}\right]\right)}{\frac{dx}{dt}} = \frac{d^2y}{dx^2}$$

$$\frac{dy}{dx} = \frac{3}{2}t$$

$$\frac{d}{dt}\left[\frac{3}{2}t\right] = \frac{3}{2}$$

$$\frac{\left(\frac{d}{dt}\left[\frac{dy}{dx}\right]\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$$

$$\frac{dy}{dx} = \left(\frac{3}{2}\right)(x+1)^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}} = \frac{\left(\frac{3}{4}\right)}{\sqrt{x+1}} = \frac{3}{4\sqrt{x+1}} = \frac{3}{4t}$$

For each additional derivative I take in parametric form (the derivative for y in terms of x), I take the derivative with respect to t of the previous derivative, and then divide the result by another x'(t).

Things that are still true when working with derivatives in parametric form:

When $\frac{dy}{dx} = 0$, the slope of the tangent line is horizontal. When $\frac{dy}{dx}$ is undefined, the slope of the tangent line may represent a cusp, or it may represent a point where the graph have a vertical tangent.

When the second derivative is 0, that is an inflection point (or undefined), which is a place where the curvature changes.

Vector-valued functions?

$$\vec{r}'(t) = \langle 2t, 3t^2 \rangle$$

If we take the derivative of a vector, we get a vector. The derivative is itself a vector: is the vector of the tangent line

In parametric form, we can create an equation of the tangent line:

$$t = 2, \vec{r}'(2) = \langle 2(2), 3(2)^2 \rangle = \langle 4,12 \rangle = \langle \Delta x, \Delta y \rangle$$

We can create a line in parametric form, but the equations $x(t) = \Delta x(t) + x_0, y(t) = \Delta y(t) + y_0$

$$\vec{T}(t) = \langle 4t + 3, 12t + 9 \rangle$$

Magnitude of the vector:

$$\|\vec{r}(t)\| = \sqrt{[x(t)]^2 + [y(t)]^2} = \sqrt{(t^2-1)^2 + (t^3+1)^2}$$

$$\|\vec{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4}$$

The magnitude of the derivative is related to the length of the curve (the arclength).

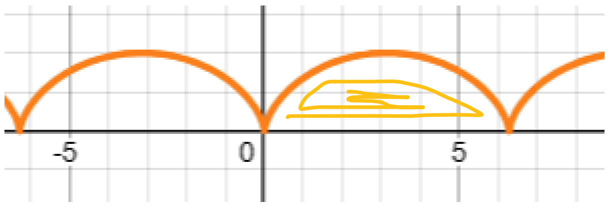
Parametric Functions and integration:

What if I want to find the area under a parametric curve?

$$A = \int_a^b y(t)x'(t)dt$$

A common parametric curve is a cycloid:

$$x(t) = t - \sin t, y(t) = 1 - \cos t, t \in [0, 2\pi]$$



$$\int_0^{2\pi} (1-\cos t)(1-\cos t)dt = \int_0^{2\pi} 1 - 2\cos t + \cos^2 t\, dt = \int_0^{2\pi} 1 - 2\cos t + \left(\frac{1}{2}\right)(1+\cos 2t)dt =$$

$$\int_0^{2\pi} \frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\, dt = \frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\Big|_0^{2\pi} = \frac{3}{2}(2\pi) = 3\pi$$

Arclength

Parametric form:

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In vector-valued function form:

$$s = \int_a^b \|\vec{r}\,'(t)\| dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

Example.
Find the arc length of the curve $x(t) = 3\cos t\,, y = 3\sin t\,, 0 \le t \le 2\pi$

$$s = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2} dt = \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt = \int_0^{2\pi} 3\sqrt{\sin^2 t + \cos^2 t}\, dt =$$

$$\int_0^{2\pi} 3\, dt = 3t|_0^{2\pi} = 6\pi$$

$$C = 2\pi r = 2\pi(3) = 6\pi$$

Example.
Find the length of arc on the curve $x(t) = t^2 - 1, y(t) = t^3 + 1$ on the interval $[0,2]$

$$\vec{r}(t) = \langle t^2 - 1, t^3 + 1\rangle, \vec{r}\,'(t) = \langle 2t, 3t^2\rangle$$

$$s = \int_0^2 \|\vec{r}\,'(t)\|\, dt = \int_0^2 \sqrt{4t^2 + 9t^4} dt = \int_0^2 \sqrt{t^2(4+9t^2)} dt = \int_0^2 t\sqrt{(4+9t^2)} dt$$

Integrate by substitution

$$u = 4 + 9t^2, du = 18tdt, \frac{1}{18}du = tdt$$

$$\int_4^{40} \frac{1}{18} u^{\frac{1}{2}} du = \frac{1}{18}\left(\frac{2}{3}u^{\frac{3}{2}}\right)_4^{40} = \frac{1}{27}\left[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] = \frac{80\sqrt{10} - 8}{27}$$

Surface areas of revolution from parametric curves (here, we will rotate around the x-axis)

$$SA = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2\pi \int_a^b y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

For rotating around the y-axis:

$$SA = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If you rotate a circle around its central axis, you get a sphere.

Find the surface area of revolution for the curve given by $(x)t = 3\cos t, y(t) = 3\sin t, t \in [0, \pi]$

$$SA = 2\pi \int_0^\pi 3\sin t \sqrt{(-3\sin t)^2 + (3\cos t)^2} dt = 2\pi \int_0^\pi 3\sin t\,(3) dt = 18\pi \int_0^\pi \sin t \, dt =$$

$$18\pi[-\cos(t)]_0^\pi = 18\pi[1 - (-1)] = 36\pi$$

$$SA = 4\pi r^2 = 4\pi(3)^2 = 36\pi$$