4/11/2024

Parametric Equations/Vectors (7.1)

Parametric equations are representations of relations/functions where both x and y are represented with expressions of a third variable, typically time.

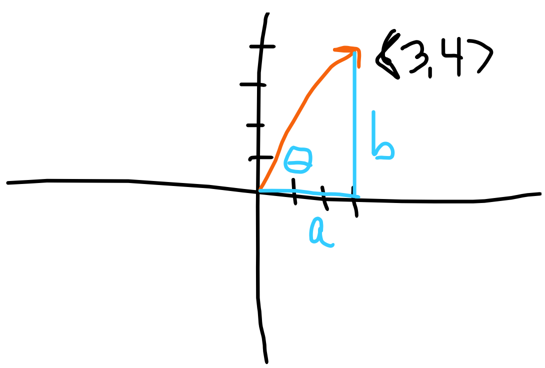
Typically, parametric equations are treated as set, but we can also represent them in vector form so that we can treat them as a single thing.

Vectors:

Are objects that have both magnitude and direction. They are often represented in coordinate form, similar to a point, with the idea that the initial point the vector starts at the origin, and the endpoint of the vector is the point/vector representation, the coordinate representation.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \langle 3,4 \rangle, 3\hat{\imath} + 4\hat{\jmath}$$

These all represent a vector that starts at the origin, and ends at the point (3,4) with the arrow at the (3,4) end.



Magnitude and direction (in a length plus angle form).
Magnitude or length of a vector $\vec{v} = \langle a, b \rangle$ is given by $\|\vec{v}\| = \sqrt{a^2 + b^2}$.
The direction is given by $\tan^{-1}\left(\frac{b}{a}\right) = \theta$ for the first and fourth quadrants and $\tan^{-1}\left(\frac{b}{a}\right) + \pi = \theta$ for the second and third quadrants.

If I have magnitude and direction form, then I can go back to coordinate form with r is the magnitude and θ the direction, with $\vec{v} = \langle r\cos\theta, r\sin\theta \rangle$

For example.
Consider the vector $\vec{u} = \langle 1,2 \rangle$ and $\vec{v} = \langle -2,-5 \rangle$

Find the magnitude of $\vec{u}$ and $\vec{v}$

$$\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\|\vec{v}\| = \sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$$

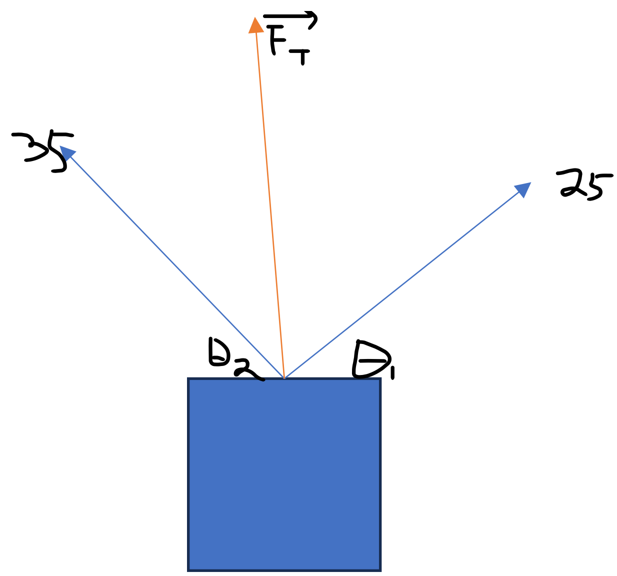
Find the direction of $\vec{u}$ and $\vec{v}$

$$\theta = \tan^{-1}\left(\frac{2}{1}\right) = 1.107 \ldots radians \approx 63.4°$$

$$\theta = \tan^{-1}\left(-\frac{5}{-2}\right) + \pi = 4.33188 \ldots radians \approx 248.2°$$

Vectors are easiest to add in coordinate form.

Suppose that two forces are being applied to an object. One force is directed at an angle of 30° to the positive horizontal axis with a force of 25 pounds. And the second force is directed at an angle of 45° to the negative horizontal axis with a force of 35 pounds. What is the resulting force magnitude and direction?



$$\vec{F}_1 = \langle 25\cos 30°, 25\sin 30°\rangle = \langle 25\left(\frac{\sqrt{3}}{2}\right), 25\left(\frac{1}{2}\right)\rangle = \langle \frac{25\sqrt{3}}{2}, \frac{25}{2}\rangle$$

$$\vec{F}_2 = \langle 35\cos(135°), 35\sin(135°)\rangle = \langle 35\left(-\frac{\sqrt{2}}{2}\right), 35\left(\frac{\sqrt{2}}{2}\right)\rangle = \langle -\frac{35\sqrt{2}}{2}, \frac{35\sqrt{2}}{2}\rangle$$

$$\vec{F}_{total} = \vec{F}_1 + \vec{F}_2 = \langle \frac{25\sqrt{3}}{2}, \frac{25}{2}\rangle + \langle -\frac{35\sqrt{2}}{2}, \frac{35\sqrt{2}}{2}\rangle = \langle \frac{25\sqrt{3}}{2} - \frac{35\sqrt{2}}{2}, \frac{25}{2} + \frac{35\sqrt{2}}{2}\rangle =$$

$$\langle -3.0981022 \ldots, 37.248737 \ldots\rangle$$

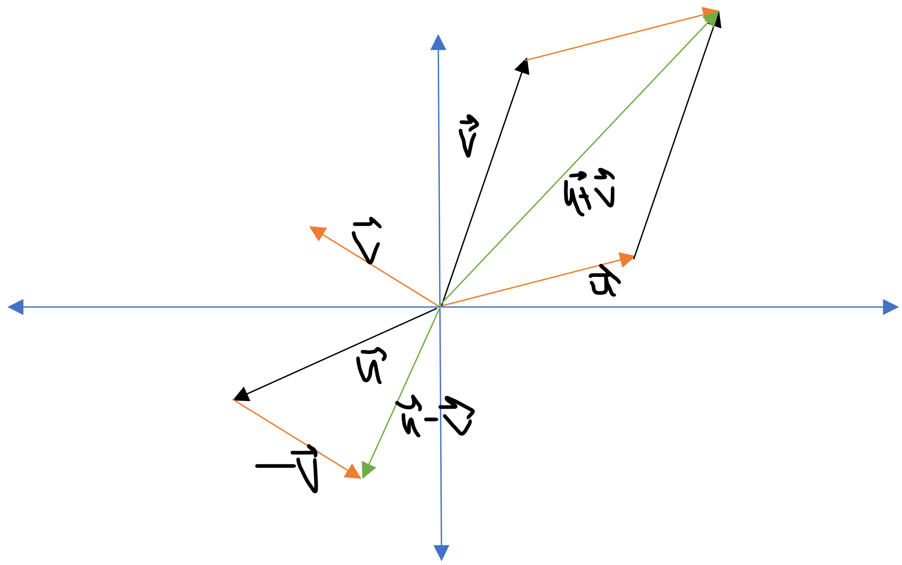
$$\left\|\vec{F}_T\right\| = \sqrt{(-3.098 \ldots)^2 + (37.248 \ldots)^2} = \sqrt{1397.06671 \ldots} \approx 37.377 \ldots$$

$$\theta = \tan^{-1}\left(\frac{37.248\ldots}{-3.098\ldots}\right) = 1.653778\ldots radians \approx 94.75°$$

So, adding vectors involves simply adding corresponding components

$$\langle 1,2\rangle + \langle -2,-5\rangle = \langle -1,-3\rangle$$

$$\langle 1,2\rangle - \langle -2,-5\rangle = \langle 3,7\rangle$$



Scale vectors:

$$3\vec{u} = 3\langle 1,2\rangle = \langle 3,6\rangle$$

Scaling a vector keeps the direction the same, but extends the magnitude.

Dot products, inner products, scalar products

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 = (1)(-2) + (2)(-5) = -12$$

Dot products are related the angle between the two vectors:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

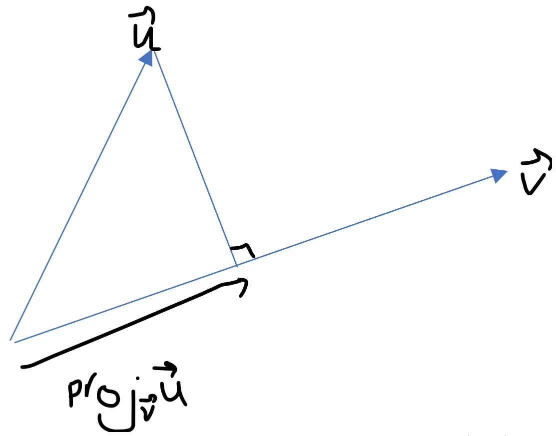
$$\cos\theta = -\frac{12}{\sqrt{5}\sqrt{29}} = 3.058\ldots \approx 175.2°$$

Negative dot products have obtuse angles between the vectors, and positive dot products have acute angles between the vectors.

If the dot product is 0, then the angle between the vector is 90-degrees, and therefore the vectors are perpendicular (orthogonal).

Project a vector u onto the vector v

$$proj_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}(\vec{v})$$



$$proj_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}(\vec{v}) = -\frac{12}{29}\langle -2,-5\rangle = \langle\frac{24}{29},\frac{60}{29}\rangle$$

$$comp_{\vec{v}}(\vec{u}) = \vec{u} - proj_{\vec{v}}(\vec{u}) = \langle 1,2\rangle - \langle\frac{24}{29},\frac{60}{29}\rangle = \langle\frac{5}{29},-\frac{2}{29}\rangle$$

Unit vector: is a vector that points in the same direction as the (original) vector, but only has a magnitude of 1. Sometimes these unit are used to represent a direction.

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||} = \langle -\frac{2}{\sqrt{29}},-\frac{5}{\sqrt{29}}\rangle$$

How does this connect to parametric equations.

Parametric form:

$$x(t) = t+2, y(t) = t^2 - 6$$

In vector form we can treat them as a unit, as components of a vector-valued equation:

$$\vec{r}(t) = \langle t+2, t^2-6\rangle$$

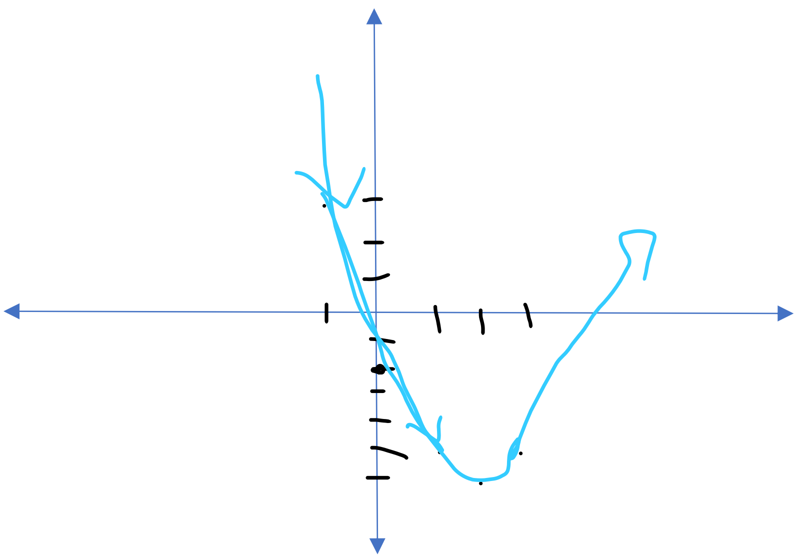
There parametric equation graphers online. Your TI-84 can also graph parametric equations.

We need to indicate the orientation of the graph, put an arrow on the curve indicating the direction of increasing time (t).

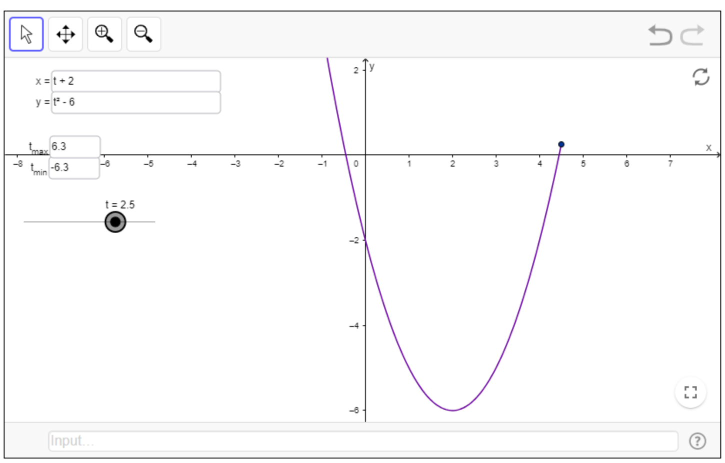
Parametrizations of functions are not unique, and the specific properties of the parametric curves will depend on the parametrization used.

Build a table of values, and then plot the curve on the graph. Add the arrow for the orientation.

t	x	y
-3	-1	3
-2	0	-2
-1	1	-5
0	2	-6
1	3	-5



https://www.geogebra.org/m/cAsHbXEU



For explicit functions of x, you can just replace x with t, and then the y component is just y(t).

Example.
Convert the function $y = x^2 + 2x - 3$ into parametric form

$$x = t, y = t^2 + 2t - 3$$
$$\vec{r}(t) = \langle t, t^2 + 2t - 3 \rangle$$

Circles and ellipses are not functions in x and y.

Circle:

$$x = r\cos t\,, y = r\sin t$$

Ellipse:

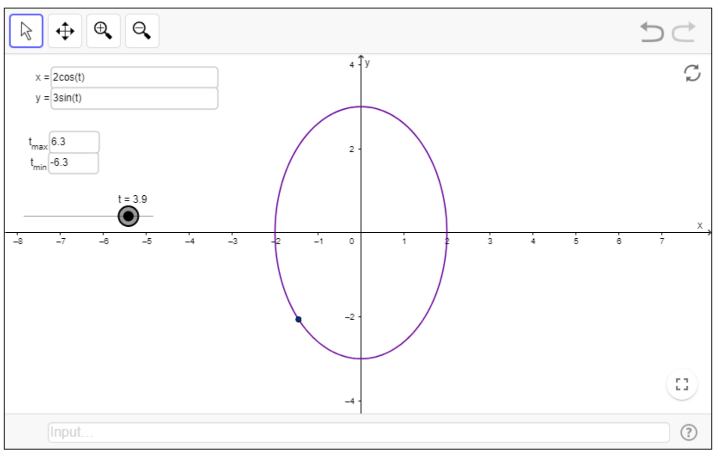
$$x = a\cos t\,, y = b\sin t$$

Example.
Find a parametric equation of the circle given by $x^2 + y^2 = 16$

$$x = 4\cos t\,, y = 4\sin t$$
$$\vec{r}(t) = \langle 4\cos t\,, 4\sin t\rangle$$

Find a parametric equation of the ellipse given by $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$$x = 2\cos t\,, y = 3\sin t$$



If I want to go in the other direction, from parametric form to rectangular (x,y) form, then in many cases, solve for t, and then substitute into the other expression. (however, if the equation is a circle or an ellipse, it's better to recognize the pattern rather than trying to substitute an inverse trig function inside trig function.)

Also, sometimes parametrizations are done in terms of other functions… so may not have to solve all the way for t, but just to some common function.

Example.

$$x = t + 2, y = t^2 - 6$$

$$x = t + 2$$
$$t = x - 2$$

$$y(x) = (x-2)^2 - 6 = x^2 - 4x + 4 - 6 = x^2 - 4x - 2$$

Example.

$$x = e^{\frac{t}{2}} + 1, y = e^t - 2$$

$$e^t = \left(e^{\frac{t}{2}}\right)^2$$

$$e^{\frac{t}{2}} = x - 1$$

$$y(x) = (x - 1)^2 - 2$$

Parametrize lines directly in parametric form more easily than standard y=mx+b form.

Consider the points (1,5), and (4,7). Find the parametrization of the line connecting the points.

Create a vector connecting the two points:

$$\langle x_2 - x_1, y_2 - y_1 \rangle = \langle \Delta x, \Delta y \rangle = \langle a, b \rangle$$

Line connecting the two points is:

$$\vec{r}(t) = \langle at + x_1, bt + y_1 \rangle$$

$$x(t) = at + x_1, y(t) = bt + y_1$$

At t=0, you are at the starting point, point 1. And at t=1, you are at the ending point, point 2.

$$\langle a, b \rangle = \langle 3,2 \rangle$$

$$x(t) = 3t + 1, y(t) = 2t + 5$$

Another advantage here is no fractions.

Next time: calculus on parametric functions.