3/7/2024

Comparison Tests (5.4)

“The Comparison Test” or “The Direct Comparison Test”

The idea here is just that if you have two series,
$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$,

If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$

And, if $a_n \geq b_n$, then $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n$

If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$, also know that $\sum_{n=1}^{\infty} b_n$ converges, then so will $\sum_{n=1}^{\infty} a_n$

$\sum_{n=1}^{\infty} b_n = L$ that implies that $\sum_{n=1}^{\infty} a_n \leq L$

if $a_n \geq b_n$, then $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n$, and we also know that $\sum_{n=1}^{\infty} b_n$ diverges, then so will $\sum_{n=1}^{\infty} a_n$

In order to prove either convergence or divergence, we need to show two things:

1) The inequality that a_n and b_n satisfy
    a. $a_n \leq b_n$ for convergence
    b. $a_n \geq b_n$ for divergence
2) Our comparison series $\sum_{n=1}^{\infty} b_n$ either converges or diverges and by what test.

Since $\sum_{n=1}^{\infty} b_n$ converges/diverges by the xxx test, we know that $\sum_{n=1}^{\infty} a_n$ converges/diverges by the (direct) comparison test.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

(in the last lecture we established that this series converges by the integral test).
Try to choose another series that is simpler than the one you have, but can be proven to converge/diverge by another test, and which has the desired inequality relationship.

The comparison here will be $\sum_{n=1}^{\infty} \frac{1}{n^2}$
(we also showed in the last lecture does converge using the integral, so we can use the p-series test here)

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, since p>1.

Show that

$$\frac{1}{n^2+1} \leq \frac{1}{n^2}$$

(As long as this inequality is true for some N>n, then it's the behavior of the tail that matters, and we can pull out the finite terms where it might not apply until the point where the inequality does hold.)

Since the numerators are the same in both cases, and the denominator is larger in the first expression than in the second, the fraction on the left will always be smaller than the one on the right.

Since $\frac{1}{n^2+1} \leq \frac{1}{n^2}$ is true, and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, we can say that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges by the direct comparison test.

Example.

$$\sum_{n=2}^{\infty} \frac{1}{n-1}$$

The series we want to compare to here is $\sum_{n=2}^{\infty} \frac{1}{n}$. And by the p-series test, we know that this series diverges, since p=1.

Then we need to establish the required inequality.

$$\frac{1}{n-1} \geq \frac{1}{n}$$

Since n-1 is always smaller than n, the smaller denominator will make the fraction on the left larger than the fraction on the right.

The series $\sum_{n=2}^{\infty} \frac{1}{n-1}$ diverges by direct comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$ which diverges by the p-series test.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

The comparison will be to $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.
This is a geometric series with $|r| < 1$ (r=1/2) and so by the geometric series test, this series will converge.

Show that:

$$\frac{1}{2^n+1} \leq \frac{1}{2^n}$$

Since $2^n + 1 \geq 2^n$, the larger number in the denominator on the left will make the fraction smaller than the one on the right.

There, by direct comparison $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ will converge because $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by the geometric series test.

Example.

$$\sum_{n=2}^{\infty}\frac{1}{\ln n}$$

Or $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ (although this can be done with the integral test, not so much the one above)

$$\ln n \le n$$

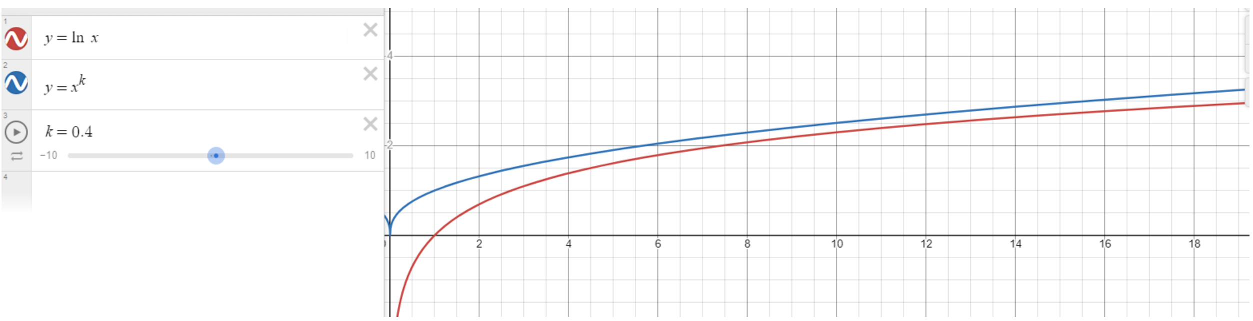
$$\frac{1}{\ln n} \ge \frac{1}{n}$$

Since the denominator is small on the left, the fraction will be bigger

By direct comparison, we can say that $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ will diverge since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-series test.

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

Try replacing ln n with n, but if that doesn't work, consider a power of n (something between 0 and 1)



The square root or 0.4 root, may make a tighter, more similar comparison.

$$\sum_{n=1}^{\infty}\frac{1}{n!}$$

Compare to the geometric series $\frac{1}{2^n}$

n	1	2	3	4	5	6	7	8	9
$\frac{1}{2^n}$	½	¼	1/8	1/16	1/32	1/64	1/128	1/256	1/512
$\frac{1}{n!}$	1	½	1/6	1/24	1/120	1/720	1/5040	1/40320	1/362880

$$\sum_{n=4}^{\infty}\frac{1}{n!} \le \sum_{n=4}^{\infty}\frac{1}{2^n}$$

And so for $n \geq 4, \frac{1}{n!} \leq \frac{1}{2^n}$, and since $\sum_{n=4}^{\infty} \frac{1}{2^n}$ converges by the geometric series, the series $\sum_{n=4}^{\infty} \frac{1}{n!}$ Converges by direct comparison.

The hard part about the direct comparison is that the inequalities have to be satisfied.

For converging series, relative to the comparison series, you can add in the denominator or subtract in the numerator, but not the reverse.

$$\frac{1}{(n^2+1)} \leq \frac{1}{n^2} \leq \frac{1}{n^2-1}$$

$$\frac{n-1}{n^3} \leq \frac{n}{n^3} \leq \frac{n+1}{n^3}$$

These are all converging series, but only the ones on the left satisfy the inequality for the direct comparison test.

The reverse is true for diverging series: we can subtract in the denominator or add in the numerator to make the series larger than a diverging series.

$$\frac{1}{n-1} \geq \frac{1}{n} \geq \frac{1}{n+1}$$

$$\frac{n+1}{n^2} \geq \frac{n}{n^2} \geq \frac{n-1}{n^2}$$

Being less than a divergent series does not show divergence directly.

The Limit Comparison test

This test compares the behavior of the series in the tail as n gets large, and if they are similar enough, the exact inequality won't matter.

If $\sum_{n=1}^{\infty} b_n$ converges/diverges, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite, non-zero value, then $\sum_{n=1}^{\infty} a_n$ also converges/diverges.

The result of infinity means a bad comparison that a_n is much larger than b_n. The result of 0 means also a bad comparison, but b_n is much bigger than a_n and you can't tell anything from that. You need a better comparison series.

The direction and location of the adding or subtracting terms in the numerator or the denominator don't matter here (the test is less sensitive to them).

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

As with the direct comparison, we'll also compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-series test.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} \times \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$$

Therefore, by the limit comparison test, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

I can make exactly the same comparison with this series using the limit comparison

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} \times \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^2}} = 1$$

Comparison series:

$$\sqrt[3]{n^4+n}$$

Use the leading term in the expression: $n^{\frac{4}{3}} = \sqrt[3]{n^4}$

Sin(n) bounded above by 1.

In a term with geometric components, those terms will be larger than any polynomial $(a^n \geq n^p)$

And take the term with the largest base in the numerator, and then separately the denominator

$$\frac{2^n + 10^n}{3^n + 5^n} \approx \frac{10^n}{5^n}$$

$$\frac{2^n + n^{10}}{3^n + n^{5000}} \approx \frac{2^n}{3^n}$$