3/5/2024

Test of Divergence (5.3)
Integral Test/P-series Test (5.3)
Alternating Series Test (5.5)

Last time we talked about geometric series and telescoping series.

Test of Divergence
Also sometimes referred to as the n-th term test.
Does not determine convergence: it can only determine if the series diverges.

If I have an infinite series $\sum_{n=1}^{\infty} a_n$, then the series will diverge if $\lim_{n\to\infty} a_n \neq 0$

If the limit of the sequence is 0, then some will converge and some will diverge, so that result is indeterminant.

This test is useful for eliminating diverging series with relatively little algebra.

Alternating Series Test applies to series where there is an alternating sign in terms $(-1)^n$ or its equivalent is part of the expression.

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

In this scenario, the AST says that the series will converge if $\lim_{n\to\infty} a_n = 0$, and will diverge otherwise.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \left(\frac{1}{1}\right) + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \frac{1}{5} + \frac{1}{6} + \cdots$$

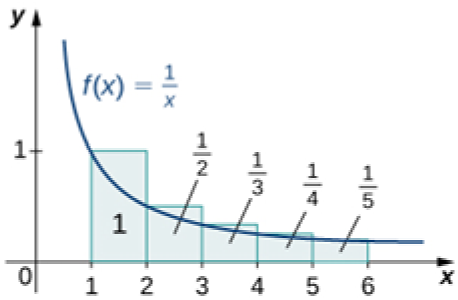
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \left(\frac{1}{2}\right) + \left(\frac{1}{12}\right) + \left(\frac{1}{30}\right) + \cdots$$

Conditional convergence: are series that converge when alternating and diverge when not alternating.

Series that converge with or without the alternating component are called absolutely convergent. They converge when $\sum_{n=1}^{\infty} |a_n|$.

Reminder: $\cos(n\pi) = (-1)^n$

Integral Test
This test says that for an infinite series $\sum_{n=1}^{\infty} a_n$, and we can express $a_n = f(n)$, then if the integral $\int_1^{\infty} f(x)dx$ converges, then the series converges, and if the integral diverges, then the series diverges.




Think of $\Delta x = 1$ as the base of our rectangles, and the height of the function is the height of the rectangle. In this illustration, they are using the left-hand rule.

Since the series acts like an estimate for that area, so if the area goes to infinity, then the series will behave similarly. If the area under the curve is finite, then the series is still an estimate to the area, but it will also be finite.

The determining factor is the behaviour of the tail of the series as n (or x) goes to infinity. If the area remains large enough long enough, the total area will be infinite and therefore diverge. If the area goes to zero quickly enough, then the area will converge and so will the series.

The area and the sum are not equal, but they will be in the same mathematical ballpark.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0, \lim_{n\to\infty} \frac{1}{n^2} = 0$$

So the nth term test is inconclusive.
Use the integral test (applies to positive function only, do not use for alternating series—can use for absolute convergence of an alternating series)

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} \ln b - \ln 1 = \infty$$

$$\int_1^{\infty} \frac{1}{x^2} dx = \int_1^{\infty} x^{-2} dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{1} = 1$$

Since this area is finite, the series also converges.

P-series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

p is a real number, generally positive
If $p \leq 1$, then the series diverges, and if p>1, then the series converges.

If p<1, we would integrate with the power rule, after we add 1, we end with a positive exponent in the numerator

p=1, we saw above, we integrate to get the natural log, and we end with the limit going to infinity.

p>1, we would integrate with the power rule, and we would still have a negative exponent after integration, infinity in the denominator, the integral and the series will converge.

$$\int_1^\infty \frac{1}{x^p}dx = \int_1^\infty x^{-p}dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \frac{1}{1-p}\lim_{b\to\infty} x^{1-p}$$

Case 1, p<1

$$\frac{1}{1-p}\lim_{b\to\infty} x^{1-p}$$

Result in a positive exponent, and therefore infinity to a positive power.

Ex. p=1/2

$$\frac{1}{1-\frac{1}{2}}\lim_{b\to\infty} x^{1-\frac{1}{2}} = 2\lim_{b\to\infty} x^{\frac{1}{2}} = \infty$$

Case 2, p>1

$$\frac{1}{1-p}\lim_{b\to\infty} x^{1-p}$$

Result in a negative exponent, and therefore infinity in the denominator
Ex. p=2
Which we saw above, or p=1.1

$$\frac{1}{1-1.1}\lim_{b\to\infty} x^{1-1.1} = -10\lim_{b\to\infty} x^{-0.1} = -10\lim_{b\to\infty}\frac{1}{x^{0.1}} = 0$$

This tail converges, and therefore both the integral and the series converges.

P-series test: if $p \leq 1$, the series diverges, and if p>1, the series converges.

Integral test, other examples.

$$\sum_{n=0}^{\infty}\frac{1}{n^2+1}, \sum_{n=1}^{\infty} ne^{-n}$$

Do these series converge or diverge?

$$\int_0^{\infty} \frac{1}{x^2+1} dx = \arctan(x)|_0^{\infty} = \lim_{b\to\infty} \arctan b - \arctan 0 = \frac{\pi}{2}$$

Converge since the integral converges.

$$\int_1^{\infty} xe^{-x} dx = -xe^{-x} - \int_1^{\infty} -e^{-x} dx = -xe^{-x} - e^{-x}|_1^{\infty} =$$

$$u = x, dv = e^{-x}dx$$
$$du = dx, v = -e^{-x}$$

$$\lim_{b\to\infty} -be^{-b} - e^{-b} + 1e^{-1} + e^{-1} = \lim_{b\to\infty} -\frac{b}{e^b} + 0 + \frac{2}{e} = \lim_{b\to\infty} -\frac{1}{e^b} + \frac{2}{e} = \frac{2}{e}$$

Converges since the integral converges.

Error calculations and estimating the number of terms needed to obtain an estimate with a given accuracy.

Both integral test and the alternating series test have error estimations.

Integral test error:

$$R_N < \int_N^{\infty} f(x)dx$$

Given that $a_n = f(n)$

Alternating series test error:

$$|R_N| < a_{N+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \ or \ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Non-alternating case, use integral test error to estimate the error on the sum of the series after 10 terms. (N=10)

$$R_{10} < \int_{10}^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{10} = \frac{1}{10} = 0.1$$

In the alternating case:
Use the alternating series error formula to estimate the error on the sum of the series after 10 terms.

$$|R_{10}| < a_{11} = \frac{1}{121} \approx 0.00826 \ldots$$

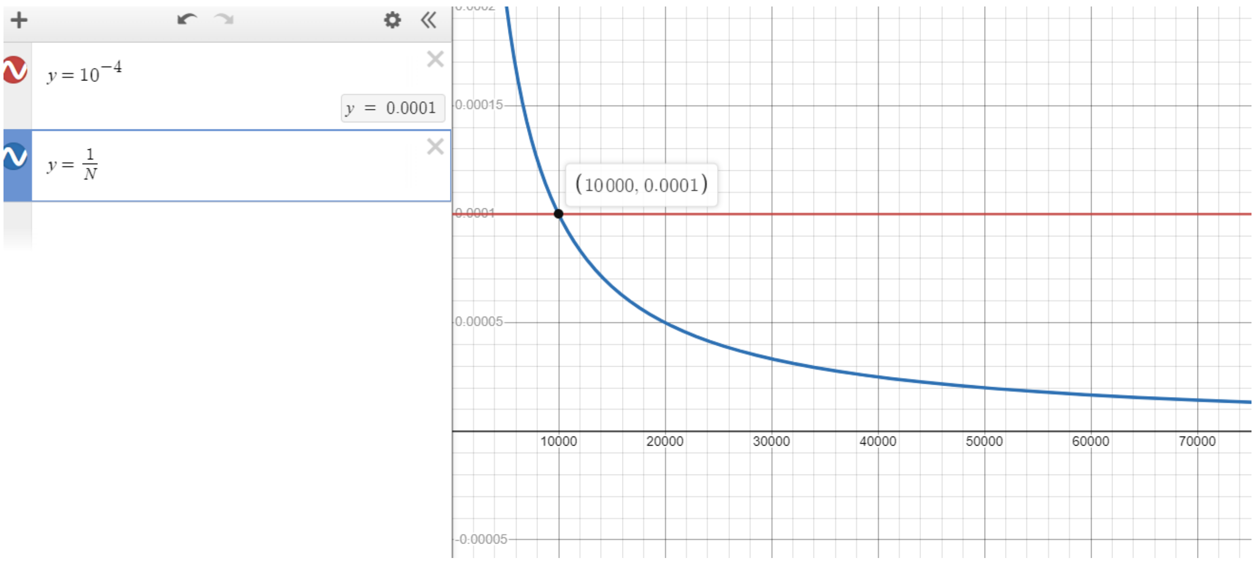
Suppose we want to find the number of terms needed to estimate the sum of the series to less than 0.0001 or $10^{-4}$.

In the non-alternating case:

$$10^{-4} < \int_N^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{N}$$

$$10^{-4} < \frac{1}{N}$$

$$N > 10^4$$

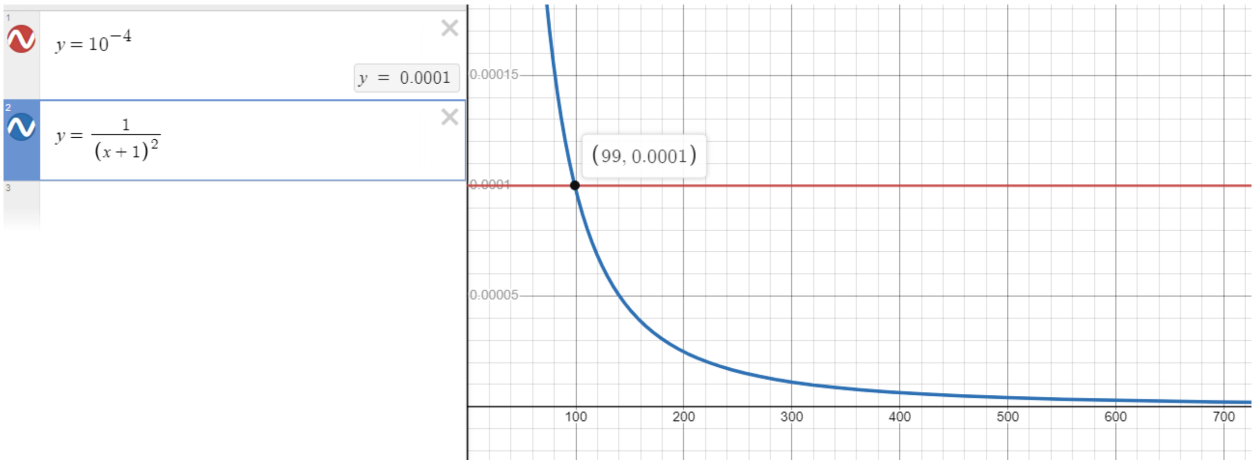



In the alternative case:

$$10^{-4} \le \frac{1}{(N+1)^2}$$

$$10^{-2} \le \frac{1}{N+1}$$

$$N + 1 \ge 100$$
$$N \ge 99$$




The n is a threshold, if you get a decimal you must round up, never down.

We have 6 infinite series tests so far:

1) Geometric series test
2) Telescoping series test
3) Divergence test/nth term test
4) Integral Test
5) P-series test
6) Alternating series test

1 and 2 have formulas for the sum if the series converges
4 and 6 have error estimates for the sum