3/28/2024

Applications of Taylor Series (6.4)
Review for Exam #2

Composition and transformation with power series.

The power series for $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Find the power series for $f(x) = \sin(x^2)$ and $g(x) = \sin\left(2x - \frac{\pi}{2}\right) = \sin\left(2\left(\mathrm{x} - \frac{\pi}{4}\right)\right)$

$$\sin\left(2x - \frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(2\left(x - \frac{\pi}{4}\right)\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)2^{2n+1}\left(x - \frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Find the power series for cos $(\sqrt{x})$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{\frac{1}{2}}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

Multiply power series
Find the power series for $f(x) = xe^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^x = \sum_{n=0}^{\infty} \frac{x(x^n)}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

We can add and subtract power series.
If we have a rational expression in factored form:

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$A(x-2)+B(x-1)=1$$
$$A+B=0$$
$$-2A-B=1$$

$$A=-B$$
$$-2A+A=1$$
$$-A=1$$
$$A=-1, B=1$$

$$\frac{1}{(x-1)(x-2)}=-\frac{1}{x-1}+\frac{1}{x-2}$$

$$\frac{1}{1-x}-\frac{1}{2-x}=\frac{1}{1-x}-\frac{\frac{1}{2}}{1-\frac{1}{2}x}=\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}x\right)^n=\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}\frac{(x)^n}{2^{n+1}}=\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)x^n$$

$$\sinh(x)=\frac{(e^x-e^{-x})}{2}=\frac{1}{2}(e^x-e^{-x})$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{-x}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$$

$$\sinh(x)=\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right]$$

$$n=0$$
$$\frac{1}{1}-\frac{1(1)}{1}=0$$

$$n=1$$
$$\frac{x}{1}-\frac{(-1)x}{1}=2x$$

$$n=2$$
$$\frac{x^2}{2}-\frac{x^2}{2}=0$$

$$n=3$$
$$\frac{x^3}{6}-\frac{(-1)x^3}{6}=\frac{2x^3}{6}$$

$$\sinh(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Consider the function:

$$f(x) = \frac{\sin(x)}{1-x}$$

(contrast with $g(x) = \sin(x)\cos(x)$)
($g(x) = \frac{1}{2}\sin(2x)$)

Division with power series

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$$x + x^2 + \frac{5}{6}x^3 + \frac{5}{6}x^4 + \cdots$$

$$1 - x \,\big)\, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$$-x + x^2$$

$$x^2 - \frac{1}{6}x^3$$

$$-x^2 + x^3$$

$$\frac{5}{6}x^3$$

$$-\frac{5}{6}x^3 + \frac{5}{6}x^4$$

$$\frac{5}{6}x^4 + \frac{x^5}{120}$$

$$-\frac{5}{6}x^4 + \frac{5}{6}x^5$$

$$\frac{101}{120}x^5$$

Consider the function $f(x) = e^x \sin(x)$
Find a power series for the function, up to degree 5 term.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

$$e^x \sin(x) \approx \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) =$$

$$x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} \ldots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{1}{30}x^5 + \cdots$$

Find the value of $\lim_{x\to 0} \frac{e^x \sin(x)}{x} = \lim_{x\to 0} \frac{x + x^2 + \frac{x^3}{3} - \frac{1}{30}x^5 + \cdots}{x} = \lim_{x\to 0} 1 + x + \frac{x^2}{3} - \frac{1}{30}x^4 + \cdots = 1$

We can also integrate and differentiate with power series

Some functions can’t be integrated in their regular form

$$\int e^{-x^2} dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int e^{-x^2} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + C$$

Use power series can be used to solve differential equations.
Last topic on Exam #2

Don’t forget about sequences…
Infinite series tests, we had 10 of them:
Geometric, telescoping, integral, p-series, alternating, direct comparison, limit comparison, nth-term/divergence test, ratio, root

Power series:
Determining convergence: interval of convergence and radius of convergence
Using an appropriate power series process for obtaining power series: geometric series formula for rational functions, and taylor series for other functions.

Taylor series have an error/remainder formula

Using power series in other applications: adding, multiplying, dividing, composing, limits, integrating (watch out for the need to re-index derivatives) – it is okay to look formulas up in the table for these specific problems.