3/21/2024

Functions as Power Series (6.1)
Convergence/Properties of Power Series (6.2)

Recall:
From the geometric series test:

$$S = \frac{a}{1-r} \leftrightarrow \sum_{n=0}^{\infty} a(r^n)$$

$$f(x) = \frac{a}{1-x} = \sum_{n=0}^{\infty} ax^n$$

Example.
Find the power series representation of the function $f(x) = \frac{5}{1-2x}$.

$$a = 5, r = 2x$$

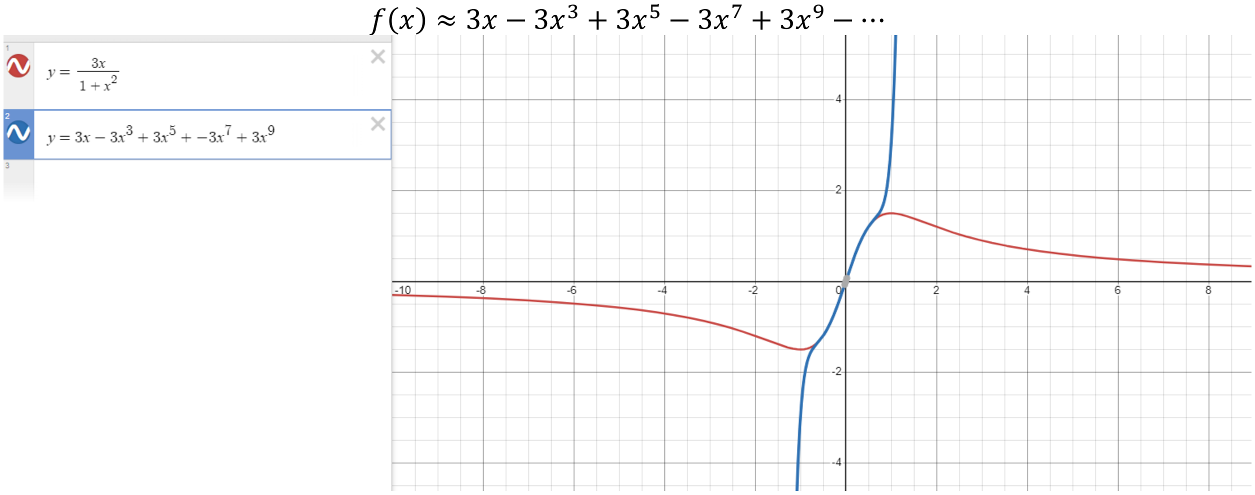
$$f(x) = \sum_{n=0}^{\infty} 5(2x)^n = \sum_{n=0}^{\infty} 5(2^n)x^n$$

Example.
Find the power series representation of the function $f(x) = \frac{3x}{1+x^2} = \frac{3x}{1-(-x^2)}$

$$a = 3x, r = -x^2$$

$$f(x) = \sum_{n=0}^{\infty} 3x(-x^2)^n = \sum_{n=0}^{\infty} (-1)^n 3x(x^{2n}) = \sum_{n=0}^{\infty} (-1)^n 3x^{2n+1}$$

$$f(x) \approx 3x - 3x^3 + 3x^5 - 3x^7 + 3x^9 - \cdots$$



Example.
Find the power series representation for the function $f(x) = \frac{7}{3-4x}$

The constant in the denominator is not 1. Multiply by something to make that constant =1.

$$\frac{7}{3-4x} \times \frac{\frac{1}{3}}{\frac{1}{3}} = \frac{\left(\frac{7}{3}\right)}{1-\frac{4}{3}x}$$

$$a = \frac{7}{3}, r = \left(\frac{4}{3}x\right)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{7}{3}\left(\frac{4}{3}x\right)^n = \sum_{n=0}^{\infty} 7\,\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^n 4^n x^n = \sum_{n=0}^{\infty} 7(4^n)3^{-n-1}x^n = \sum_{n=0}^{\infty} \frac{7(4^n)}{3^{n+1}}x^n$$

Example.
Find the power series representation of the function $f(x) = \frac{4}{3x-1}$

$$\frac{4}{3x-1} \times \frac{-1}{-1} = -\frac{4}{1-3x}$$

$$a = -4, r = 3x$$

$$f(x) = \sum_{n=0}^{\infty}(-4)(3x)^n = \sum_{n=0}^{\infty}(-4)3^n x^n$$

So far, all of our examples have been centered at c=0.
We can expand our power series at points other than 0.

Example.
Find the power series representation of the function $f(x) = \frac{1}{x}$

$$\frac{1}{x} = \frac{1}{1-1+x} = \frac{1}{1-(1-x)}$$

If I center the power series at another point, other than 0, I can replace x in the power series with a linear expression in x, (x-c)

$$a = 1, r = (1-x) = (-1)(x-1)$$

$$f(x) = \sum_{n=0}^{\infty} 1[(-1)(x-1)]^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$


Example.

Find the power series representation of the function $f(x) = \frac{5}{2-3x}$, centered at $x = 4$.

$$\frac{5}{2-3x} = \frac{5}{2-3(x-4)-12} = \frac{5}{2-3x+12-12} = \frac{5}{-10-3(x-4)} \times \frac{\left(-\frac{1}{10}\right)}{\left(-\frac{1}{10}\right)} = \frac{\left(-\frac{1}{2}\right)}{1+\frac{3}{10}(x-4)}$$

$$a = -\frac{1}{2}, r = -\frac{3}{10}(x-4)$$

$$f(x) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)\left[\left(-\frac{3}{10}\right)(x-4)\right]^n = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{3}{10}\right)^n (x-4)^n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}3^n}{2\cdot 10^n}(x-4)^n$$

We can adjust the constant in the denominator to match the formula (1- something)

We can shift the center off of zero

We can fix the minus sign if we have (1+ something)

We can also find power series formulas for functions whose derivatives are rational expressions.

These include $f(x) = \ln x, g(x) = \arctan x$

First take the derivative of the target function: $g'(x) = \frac{1}{1+x^2}$

Then, find a power series representation for the derivative

Then, take the antiderivative to obtain the power series for the original function.

$$a = 1, r = -x^2$$

$$g'(x) = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\int g'(x)dx = g(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Additional formulas can be obtained by taking the derivative of our initial power series formula.

$$a(1-r)^{-1} = \frac{a}{1-r} = \sum_{n=0}^{\infty} a(r^n)$$

Take the derivative on both sides with respect to r.

$$a(-1)(1-r)^{-2}(-1) = \frac{a}{(1-r)^2} = \sum_{n=1}^{\infty} anr^{n-1} = \sum_{k=0}^{\infty} a(k+1)r^k$$

Reindexing: replacing n with k+1

$$a(-2)(1-r)^{-3}(-1) = \frac{2a}{(1-r)^3} = \sum_{n=2}^{\infty} an(n-1)r^{n-2} = \sum_{k=0}^{\infty} a(k+2)(k+1)r^k$$

$$2a(-3)(1-r)^{-4}(-1) = \frac{6a}{(1-r)^4} = \sum_{n=3}^{\infty} an(n-1)(n-2)r^{n-3} = \sum_{k=0}^{\infty} a(k+3)(k+2)(k+1)r^k$$

And so on… and each time reindex the starting value to start at 0.
Replace n = k+2
Replace n=k+3

Example.
Find a power series representation for the function $f(x) = \frac{7x}{\left(1-\frac{1}{2}x^2\right)^3}$

Notice that the denominator is raised (all of it) to a cube power.
The derivative is for a rational expression with the entire denominator raised to the cube power.

$$\frac{2a}{(1-r)^3} = \sum_{k=0}^{\infty} a(k+2)(k+1)r^k$$

$$2a = 7x, a = \frac{7}{2}x, r = \frac{1}{2}x^2$$

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{7}{2}x\right)(k+2)(k+1)\left(\frac{1}{2}x^2\right)^k = \sum_{k=0}^{\infty} 7\left(\frac{1}{2}\right)^{k+1} x(k+2)(k+1)x^{2k} =$$
$$\sum_{k=0}^{\infty} 7\left(\frac{1}{2}\right)^{k+1} (k+2)(k+1)x^{2k+1}$$

Example.

Find the power series representation of the function $f(x) = \frac{x^3}{(2-x)^2}$

$$\frac{x^3}{(2-x)^2} \times \frac{\frac{1}{4}}{\frac{1}{4}} = \frac{\left(\frac{x^3}{4}\right)}{(2-x)^2\left(\frac{1}{2}\right)^2} = \frac{\left(\frac{x^3}{4}\right)}{\left(\frac{2-x}{2}\right)^2} = \frac{\left(\frac{x^3}{4}\right)}{\left(1-\frac{1}{2}x\right)^2}$$

$$\frac{a}{(1-r)^2} = \sum_{k=0}^{\infty} a(k+1)r^k$$

$$a = \left(\frac{x^3}{4}\right), r = \frac{1}{2}x$$

$$f(x) = \sum_{k=0}^{\infty} \frac{x^3}{4}(k+1)\left(\frac{1}{2}x\right)^k = \sum_{k=0}^{\infty} x^3\left(\frac{1}{2}\right)^2 (k+1)\left(\frac{1}{2}\right)^k x^k = \sum_{k=0}^{\infty} (k+1)\left(\frac{1}{2}\right)^{k+2} x^{k+3}$$

The terms that go into the power series must (!!!!!) be x or (x-c), they cannot be general polynomials.

Example.

Sometimes this means you may have to complete the square.

Find the power series representation of the function $f(x) = \frac{1}{1+4x+x^2}$

You cannot let $r = -4x - x^2$

$$\frac{1}{1+4x+x^2} = \frac{1}{1-4+(x^2+4x+4)} = \frac{1}{-3+(x+2)^2} \times \frac{-\frac{1}{3}}{-\frac{1}{3}} = \frac{-\frac{1}{3}}{1-\frac{1}{3}(x+2)^2}$$

$$a = -\frac{1}{3}, r = \frac{1}{3}(x+2)^2$$

$$f(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)\left[\left(\frac{1}{3}\right)(x+2)^2\right]^n = \sum_{n=0}^{\infty} -\frac{(x+2)^{2n}}{3^{n+1}}$$

Testing for convergence of power series:

See the end of the last class notes for finding the interval and radius of convergence.