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Review of series tests to date
Ratio and Root Tests

So, far, we have 8 series test:

1) Geometric series test (if it converges, we can find the exact sum)
2) Telescoping series test (if it converges, we can find the exact sum)
3) Divergence test – only tests for divergence, not convergence
4) Integral test – (we can estimate the error on the sum after N terms)
5) P-series test – powers of n in the denominator
6) Alternating series test – (we can estimate the error on the sum after N terms)
7) Direct Comparison Test
8) Limit Comparison Test

The last two series test that we have to cover are the ratio and root tests.

Ratio Test
Given the infinite series $\sum_{n=0}^{\infty} a_n$, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series will converge, and if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, the series will diverge, and if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive.

If $a_n = r^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{r^{n+1}}{r^n}\right| = |r|$<1, then the series converges, and $r \geq 1$ the series diverges.

Consider $\sum_{n=1}^{\infty}\frac{1}{n}$ vs. $\sum_{n=1}^{\infty}\frac{1}{n^2}$

We know from the integral test that the harmonic series diverges, and the second one (a p-series) converges.

$$\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

$$\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1} = \lim_{n\to\infty}\frac{2n}{2n+2} = \lim_{n\to\infty}\frac{2}{2} = 1$$

Typically, the ratio test does poorly with polynomial or rational function terms.
Does a good job with anything raised to a power of n, and factorials.

Examples.

$$\sum_{n=1}^{\infty}\frac{n^3}{3^n}$$

$$\lim_{n\to\infty}\frac{\left(\frac{(n+1)^3}{3^{n+1}}\right)}{\frac{n^3}{3^n}}=\lim_{n\to\infty}\left[\frac{(n+1)^3}{3^{n+1}}\times\frac{3^n}{n^3}\right]=\lim_{n\to\infty}\frac{(n+1)^3}{n^3}\times\lim_{n\to\infty}\frac{3^n}{3^{n+1}}=$$

$$\lim_{n\to\infty}\frac{n^3+3n^2+3n+1}{n^3}\times\lim_{n\to\infty}\frac{3^n}{3^n(3)}=\lim_{n\to\infty}\frac{\frac{n^3}{n^3}+\frac{3n^2}{n^3}+\frac{3n}{n^3}+\frac{1}{n^3}}{\frac{n^3}{n^3}}\times\lim_{n\to\infty}\frac{1}{3}=$$

$$\lim_{n\to\infty}\frac{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}}{1}\times\left(\frac{1}{3}\right)=1\times\frac{1}{3}=\frac{1}{3}<1$$

Example.

$$\sum_{n=0}^{\infty}\frac{2^n}{n!}$$

$$\lim_{n\to\infty}\left(\frac{2^{n+1}}{(n+1)!}\right)\times\frac{n!}{2^n}=\lim_{n\to\infty}\frac{n!}{(n+1)!}\times\frac{2^{n+1}}{2^n}=\lim_{n\to\infty}\frac{n!}{n!\,(n+1)}\times\frac{2^n(2)}{2^n}=\lim_{n\to\infty}\frac{2}{n+1}=0$$

Converges by the ratio test.

Example.

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{(n+1)}[(n+1)!]^2}{\big(2(n+1)\big)!}\times\frac{(2n)!}{(-1)^n(n!)^2}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n(-1)(n+1)!\,(n+1)!}{(2n+2)!}\times\frac{(2n)!}{(-1)^n(n!)(n!)}\right|=$$

$$\lim_{n\to\infty}\left|\frac{(n+1)n!\,(n+1)n!}{(2n+2)(2n+1)(2n)!}\times\frac{(2n)!}{(n!)(n!)}\right|=\lim_{n\to\infty}\left|\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\times\frac{1}{1}\right|=\frac{1}{4}<1$$

Converges

Example.

$$\sum_{n=1}^{\infty}\frac{n^n}{n!}$$

$$\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\times\frac{n!}{n^n}=\lim_{n\to\infty}\frac{(n+1)^n(n+1)}{(n+1)n!}\times\frac{n!}{n^n}=\lim_{n\to\infty}\frac{(n+1)^n}{1}\times\frac{1}{n^n}=\lim_{n\to\infty}\frac{(n+1)^n}{n^n}=$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = L$$

$$\lim_{n\to\infty} ln\left(1+\frac{1}{n}\right)^n = \ln L = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{n^{-1}} = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \frac{0}{0}$$

$$\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

$$1 = \ln L \to L = e$$

$e > 1$, so the series diverges

Root Test

Given the series $\sum_{n=1}^{\infty} a_n$, if the $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$ the series converges, and if $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} > 1$, the series diverges. And if $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1$ the test is inconclusive.

$$\lim_{n\to\infty}\sqrt[n]{n} = 1$$

Consider $\sum\frac{1}{n}$ and $\sum\frac{1}{n^2}$

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{n}} = 1$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{n}\right)^2} = 1$$

The root test also tends to be inconclusive with rational and polynomial terms.
Root test is messy to use on factorials – you would need to use a replacement approximation that relates factorials to an exponential expression. I recommend using the ratio test for anything that has a factorial in it.
Geometric combined with polynomial components, or expressions raised to a common power of n are the best for the root test.

Example.

$$\sum_{n=1}^{\infty}\frac{n^3}{3^n}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^3}{\sqrt[n]{3^n}} = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^3}{3} = \frac{1}{3} < 1$$

Converges by the root test.

Example.

$$\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n} = \sum_{n=1}^{\infty} \left(\frac{n^2+3n}{4n^2+5}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+3n}{4n^2+5}\right)^n} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4} < 1$$

Converges by the root test.

Example.

$$\sum_{n=2}^{\infty} \frac{n^n}{(\ln n)^n}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$$

Diverges by the root test.

Power Series
Infinite series where there is an x raised to the nth power in the expression.
For what values of x does the series converge?

Example.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = \lim_{n\to\infty} \frac{x^n(x)}{(n+1)n!} \times \frac{n!}{x^n} = \lim_{n\to\infty} \frac{1(x)}{(n+1)} \times \frac{1}{1} = 0$$

In the limit, think about x as any fixed value, and so n will (eventually) be bigger than x and the limit will go to 0.

Where does this converge? It converges for all real numbers, and the radius of convergence here is infinity.

Example.

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{(n+1)3^n}$$

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}\times\frac{(n+1)3^n}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^n(x-2)}{(n+2)3^n(3)}\times\frac{(n+1)3^n}{(x-2)^n}\right| =$$

$$\lim_{n\to\infty}\left|\frac{(x-2)}{(n+2)(3)}\times\frac{(n+1)}{1}\right| = \lim_{n\to\infty}\left|\frac{(x-2)}{3}\times\frac{(n+1)}{(n+2)}\right| = \lim_{n\to\infty}\left|\frac{(x-2)}{3}\times 1\right| < 1$$

$$\left|\frac{x-2}{3}\right| < 1$$

$$-1 < \frac{x-2}{3} < 1$$

$$-3 < x-2 < 3$$

Radius of convergence: 3

$$-1 < x < 5$$

Interval of convergence is (-1,5) … so far.

If the series converges on an interval (a,b), the radius of convergence is $\frac{b-a}{2}$

$$\frac{5-(-1)}{2} = \frac{6}{2} = 3$$

We need to test the endpoints where the ratio test =1 by another test.
Test x= -1, and x=5

Check x= -1

$$\sum_{n=1}^{\infty}\frac{(-1-2)^n}{(n+1)3^n} = \sum_{n=1}^{\infty}\frac{(-3)^n}{(n+1)3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(3)^n}{(n+1)3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)}$$

This converges by the alternating series test.

Check x=5

$$\sum_{n=1}^{\infty}\frac{(5-2)^n}{(n+1)3^n} = \sum_{n=1}^{\infty}\frac{(3)^n}{(n+1)3^n} = \sum_{n=1}^{\infty}\frac{1}{(n+1)}$$

By the limit comparison test

$$\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

They converge or diverge together and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test (or the integral test), so this also diverges.

The final interval of convergence is $[-1,5)$

It is possible to have intervals of convergence that are open on both ends (a,b), or closed on one end and not the other (a,b], or [a,b), or converge on both ends [a,b].

Typically depends if there is an extra n term in the denominator:
No n means both endpoints will diverge (or in the numerator)
One n means one but not the other will converge
n^2 or higher, then both ends will converge