2/29/2024

Introduction to infinite series
Geometric and telescoping series

Series vs. sequence
A series is the sum of a sequence.
a_n is the sequence notation
$\sum_{n=1}^{\infty} a_n$ is the series (infinite series)

A finite series $\sum_{n=1}^{N} a_n$ it is the sum of a finite number of terms of a sequence

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N = S_N$$

S_N is a partial sum of the series

Find the 6th partial sum (S_6) of the series $\sum_{n=1}^{\infty} \frac{1}{n}$

$$S_1 = \frac{1}{1} = 1$$

$$S_2 = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

$$S_4 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$$S_5 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$$

$$S_6 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{49}{20}$$

A sequence of partial sums:

$$S_k = \left\{1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \ldots\right\}$$

If the sequence of partial sums has a limit, then the infinite series has a sum.

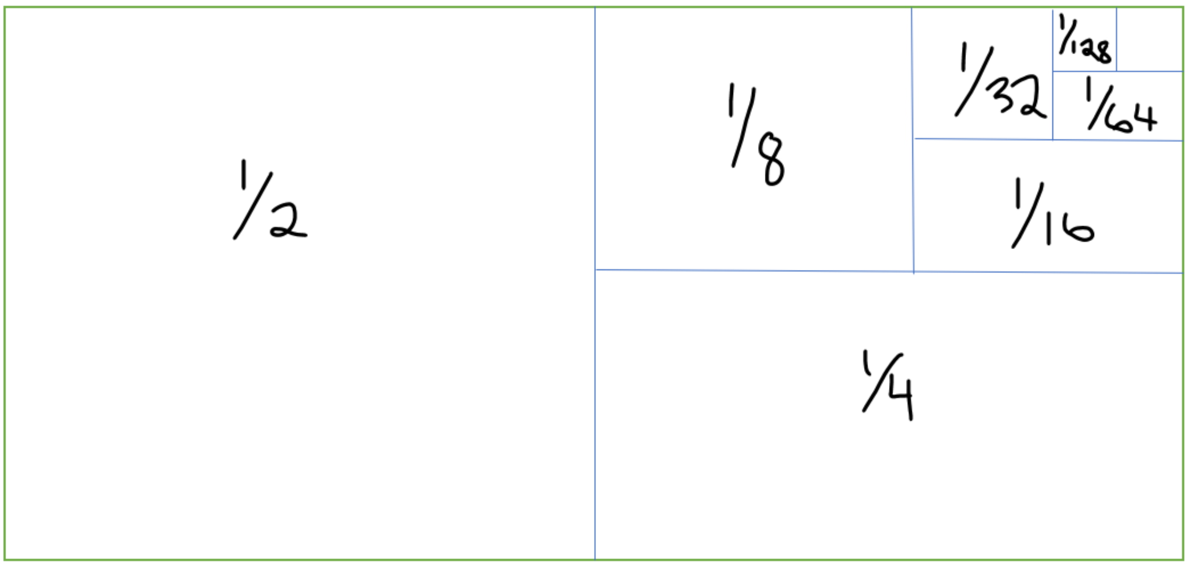
Geometric series

$$\sum_{n=0}^{\infty} a(r^n)$$

If the sum exists and is finite, we say that the series converges
If the sum doesn't exist, or it goes to infinity, we say the series diverges.

$|r| < 1$, the sum of the series exists, series converges
If $|r| \geq 1$, the series diverges

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$$



r is common ratio, if that is |r|<1, then the series converges.

There is a formula for the sum of an infinite geometric series.

$$S_\infty = \sum_{n=0}^{\infty} a(r^n) = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1 = \frac{1}{1-\frac{1}{2}} - 1 = \frac{1}{\frac{1}{2}} - 1 = 2 - 1 = 1$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Example.
Find the sum of infinite geometric series

$$\sum_{n=0}^{\infty} 5\left(-\frac{2}{7}\right)^n = \frac{5}{1-\left(-\frac{2}{7}\right)} = \frac{5}{1+\frac{2}{7}} = \frac{5}{\frac{9}{7}} = 5 \times \frac{7}{9} = \frac{35}{9}$$

$$\sum_{n=0}^{\infty} 5\left(\frac{2}{7}\right)^n = \frac{5}{1-\left(\frac{2}{7}\right)} = \frac{5}{\frac{5}{7}} = 5 \times \frac{7}{5} = 7$$

Example.

$$\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1 = \infty$$

$$\sum_{n=0}^{\infty} (-1)^n$$

$$S_n = \{1, 1+(-1) = 0, 1, 0, 1, 0 \ldots\}$$

There is no limit to this sequence and so it can't converge

Repeating decimal applications for geometric series.

$$0.43434343 \ldots. = 0.\overline{43}$$
$$\frac{43}{99} = 0.43434343 \ldots$$

$$0.215215215 \ldots = 0.\overline{215}$$
$$\frac{215}{999} = 0.215215215 \ldots$$

Convert, using geometric series properties, 0.434343… to a fraction.

Rewrite as a geometric series.

$$0.43 + 0.0043 + 0.000043 + \cdots$$
$$43(0.01 + 0.0001 + 0.000001 + \cdots)$$
$$43\left(\frac{1}{100} + \frac{1}{10{,}000} + \frac{1}{1{,}000{,}000} + \cdots\right) = 43(10^{-2} + 10^{-4} + 10^{-6} + \cdots) =$$

$$43[(10^{-2})^1 + (10^{-2})^2 + (10^{-2})^3 + \cdots] = \frac{43}{100}[(10^{-2})^0 + (10^{-2})^1 + (10^{-2})^2 + \cdots] =$$

$$\frac{43}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n = \frac{\frac{43}{100}}{1-\frac{1}{100}} = \frac{\frac{43}{100}}{\frac{99}{100}} = \frac{43}{100} \times \frac{100}{99} = \frac{43}{99}$$

Infinite (or finite ones) follow rules of addition…

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Telescoping series

Is a series that has the property that subsequent terms in the series will cancel out previous terms in the series, so that the final sum depends only on initial terms and ending terms.

Typically, the terms in the series (sequences) have the form $\frac{1}{(n)(n+k)}$

Ex. $\frac{1}{n(n+1)}, \frac{1}{n(n+2)}, \frac{1}{(n+1)(n+4)}, \frac{1}{(2n+1)(2n+3)}, \cdots$

Apply partial fractions to rewrite the problem in the form $\frac{A}{n} - \frac{A}{n+1}, \frac{A}{n} - \frac{A}{n+2}, etc.$

You will end up with something like:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots =$$

$$1 - \lim_{n\to\infty}\frac{1}{n+1} = 1$$

Example.

$$\sum_{n=1}^{\infty}\frac{4}{n^2 + 2n} = \sum_{n=1}^{\infty}\frac{4}{n(n+2)} =$$

$$\frac{4}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2) + B(n)}{n(n+2)}$$

$$An + 2A + Bn = 4$$

$$A + B = 0$$
$$2A = 4, A = 2, B = -2$$

$$\sum_{n=1}^{\infty}\frac{4}{n(n+2)} = \sum_{n=1}^{\infty}\frac{2}{n} - \frac{2}{n+2} =$$

$$\left(\frac{2}{1} - \frac{2}{3}\right) + \left(\frac{2}{2} - \frac{2}{4}\right) + \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{4} - \frac{2}{6}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) + \cdots =$$

$$\frac{2}{1} + \frac{2}{2} + \cdots - \lim_{n\to\infty}\left(\frac{2}{n+1} + \frac{2}{n+2}\right) = 3$$

converges

Example.

$$\sum_{n=1}^{\infty} \frac{\ln n}{\ln(n+1)} = \sum_{n=1}^{\infty} [\ln n - \ln(n+1)] = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots =$$

$$\ln 1 - \lim_{n\to\infty} \ln(n+1) = -\infty$$

Diverges

Geometric and telescoping series (tests), they are the only ones where you can calculate the sum. For most series in this chapter, we will only care if the series converges or diverges because for most series that's all we can do. But these two types, we can calculate the actual sum.