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Sequences (starting Chapter 5 in the online book)

A sequence is an ordered list of values.

$$\{1, 2, 4, 8, 16, 32, \dots\}$$

$$a_n = \{a_0, a_1, a_2, \dots\}$$

$$a_0 = 1, a_1 = 2, a_5 = 32, a_6 = 64$$

$$a_n = 2^n$$

An explicit formula is a function of the form $a_n = f(n)$ that takes you directly to any point in the sequence without all the intermediate steps being calculated.

Recursive formula requires you to calculate consecutive values in the sequence to obtain further values, the formula is based on past values.

$$a_n = a_{n-1}(2)$$
$$a_6 = a_5(2)$$

Fibonacci sequence:

$$a_n = a_{n-1} + a_{n-2}$$

Recursive formulas also require a seed… to tell you where to start.
If the recursive formulas depends on only one previous term then you need only one seed value, if it depends on two previous values, then we need two seeds.

$$a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13, \dots$$

You may be asked to write out the first couple of terms of the sequence based on either recursive or an explicit formula.

Example. Write out the first 5 terms of the sequence given by $a_n = 3 + a_{n-1}, a_0 = 1$

$$a_0 = 1, a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_5 = 16, \dots$$

Arithmetic sequence, the terms in the sequence differ by a common difference.
Geometric sequences have terms where they differ by a common ratio.

Example. Write out the first 5 terms of the sequence given by $a_n = 1 + 3n$

$$a_0 = 1, a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_5 = 16, \dots$$

Writing a given sequence as a formula:
Identify the pattern in the sequence and then express that as an explicit function (whenever possible)

Example.

$$\left\{\frac{3}{2},\frac{7}{4},\frac{15}{8},\frac{31}{16},\frac{63}{32},\ldots\right\}$$

1) Find a pattern in the numerator and the denominator separately.
2) Compare to common patterns you already know

Denominator: 2, 4, 8, 16, 32… 2^n
Numerator: 3, 7, 15, 31, 63, … (4-1), (8-1), (16-1), (32-1), (64-1) … $2^k - 1 = 2^{n+1} - 1$

$$a_n = \frac{2^{n+1}-1}{2^n}$$

(note: start at n=1)

Starting at n=0: $a_n = \frac{2^{n+2}-1}{2^{n+1}}$

Exponential/Geometric:

$$2, 4, 8, 16, 32, \ldots$$
$$1, 3, 9, 27, 81, \ldots$$

Arithmetic:

$$1, 4, 7, 10, 13, \ldots$$
$$2, 4, 6, 8, 10, \ldots$$
$$1, 3, 5, 7, 9, \ldots$$

Powers:

$$1, 4, 9, 16, 25, \ldots$$
$$1, 8, 27, 64, 125, \ldots$$

Factorial:

$$1, 1, 2, 6, 24, 120, 720, \ldots$$

$$n! = n(n-1)(n-2)\ldots.(3)(2)(1)$$

$$4! = 4(3)(2)(1) = 24$$
$$7! = 7(6)(5)(4)(3)(2)(1) = 5040$$

$$0! = 1$$

If the sign changes $(-1)^n$

Example: write an explicit formula for the sequence:

$$\left\{1, 2x, 2x^2, \frac{4}{3}x^3, \frac{2}{3}x^4, \frac{4}{15}x^5 \ldots\right\}$$

$$\left\{\frac{1}{1}, \frac{2x}{1}, \frac{2x^2}{1}, \frac{4}{3}x^3, \frac{2}{3}x^4, \frac{4}{15}x^5 \ldots\right\}$$

$$\left\{\frac{1}{1}, \frac{2x}{1}, \frac{4x^2}{2}, \frac{8}{6}x^3, \frac{16}{24}x^4, \frac{32}{120}x^5 \ldots\right\}$$

$$a_n = \frac{2^n x^n}{n!}, at\ n = 0$$

$$\left\{\frac{2^n x^n}{n!}\right\}_{n=0}^{\infty} \leftrightarrow \left\{\frac{2^n x^n}{n!}\right\}_{0}^{\infty}$$

Where does the sequence go?

What is the limit of the sequence as n goes to infinity? Is there a limit of the sequence and if so, what is it?

$$a_n = f(n)$$

Think of the sequence as a function of n.

n is a non-negative integer (whole number, or counting numbers give or take 0)
As n gets very large, is there some value that my function is approaching.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n)$$

Example. Find the limit of the sequence given by $a_n = 1 - \left(\frac{1}{2}\right)^n$

$$f(n) = 1 - \left(\frac{1}{2}\right)^n$$

$$\lim_{n\to\infty} 1 - \left(\frac{1}{2}\right)^n = 1 - \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 1 - 0 = 1$$

Not all sequences have limits that are defined.

$$\lim_{n\to\infty} (-1)^n = DNE$$

$$\{1, -1, 1, -1, 1, -1, 1 \ldots\}$$

This sequence "diverges" because there is no limit.

Squeeze Theorem

Example. Find the limit of the sequence $a_n = \frac{\sin(n)}{n}$

$$\lim_{n\to\infty} \frac{\sin(n)}{n}$$

$$-1 \leq \sin(n) \leq 1$$

Sine is bounded between -1 and 1 for all n.

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

$$\lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{\sin(n)}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

$$0 \leq \lim_{n\to\infty} \frac{\sin(n)}{n} \leq 0$$

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

Example.
Finding the limit of the sequence $\left\{\frac{3n^4-7n^2+5}{6-4n^4}\right\}$

$$\lim_{n\to\infty} \frac{3n^4-7n^2+5}{6-4n^4} = \lim_{n\to\infty} \frac{\frac{3n^4}{n^4}-\frac{7n^2}{n^4}+\frac{5}{n^4}}{\frac{6}{n^4}-\frac{4n^4}{n^4}} = \lim_{n\to\infty} \frac{3-\frac{7}{n^2}+\frac{5}{n^4}}{\frac{6}{n^4}-4} = \frac{3-0-0}{0-4} = -\frac{3}{4}$$

$$\lim_{n\to\infty} \frac{3n^4-7n^2+5}{6-4n^4} = \lim_{n\to\infty} \frac{12n^3-14n}{-16n^3} = \lim_{n\to\infty} \frac{36n^2-14}{-48n^2} = \lim_{n\to\infty} \frac{72n}{-96n} = \lim_{n\to\infty} \frac{72}{-96} = -\frac{3}{4}$$

Sometimes you'll have expressions disguised as something more complex than absolutely necessary.

$$\cos(n\pi) = (-1)^n$$

$$\sin\left(\frac{2n+1}{2}\pi\right) = (-1)^n$$

Bounded and monotonic sequences

Bounded sequences are contained between two finite values (neither go to infinity or negative infinity).
Bounded below means there is a minimum value for the sequence that it never goes below.
Bounded above means there is a maximum value for the sequence that it never goes above.

You don't have to consider negative values for n, only for $n \geq 0$.

Monotonic just means it's always increasing or always decreasing (doesn't change direction)
May treat a function as monotonic if it is monotonic after a finite number of terms.

Bounded and monotonic sequence have limits.
A sequence that is both bounded and monotonic must have a limit (even if you don’t know what it is).

Pick up with infinite series.