2/15/2024

Integration by Tables
Numerical Integration

Integration by tables in this class should be used when you are told to use integration by tables. All other integration problems should be done with the integration techniques we learned in the rest of the chapter. (I need to see the work.)

Example.

$$\int \frac{\ln|x| \arctan|\ln|x||}{x} dx$$

$$u = \ln|x|, du = \frac{1}{x} dx$$

$$\int u \arctan u \, du$$

(5.7) $$\int x \arctan x dx = \frac{1}{2} x^2 \arctan x + \frac{1}{2} \arctan x - \frac{1}{2} x + C$$

$$= \frac{1}{2} u^2 \arctan u + \frac{1}{2} \arctan u - \frac{1}{2} u + C =$$

$$\frac{1}{2} \ln^2 |x| \arctan(\ln|x|) + \frac{1}{2} \arctan(\ln|x|) - \frac{1}{2} \ln|x| + C$$

Example.

$$\int \frac{3x}{2x+7} dx = 3 \int \frac{x}{2x+7} dx$$

(2.6) $$\int \frac{x}{ax+b} dx = \frac{x}{a} - \frac{b}{a^2} \ln|ax+b| + C$$

$$a = 2, b = 7$$

$$= 3\left[\frac{x}{2} - \frac{7}{4} \ln|2x+7|\right] + C$$

Example.

$$\int \frac{1}{1 - \cos(4x)} dx$$

(4.32) $$\int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot\left(\frac{ax}{2}\right) + C$$

$$a = 4$$

$$= -\frac{1}{4}\cot(2x) + C$$

Example.

$$\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx$$

$$u = e^x, du = e^x dx$$

$$\int \frac{1}{\sqrt{u^2 - 4}} du$$

(3.35) $$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

$$a = 2$$

$$= \ln\left|u + \sqrt{u^2 - 4}\right| + C = \ln\left|e^x + \sqrt{e^{2x} - 4}\right| + C$$

Example.

$$\int \frac{\cos x}{\sin^2 x + 2\sin x} dx$$

$$u = \sin x\, , du = \cos x\, dx$$

$$\int \frac{1}{u^2 + 2u} du = \int \frac{1}{u(u+2)} du$$

(2.17) $$\int \frac{1}{x(ax+b)} dx = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$$

$$a = 1, b = 2$$

$$= \frac{1}{2}\ln\left|\frac{u}{u+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sin x}{\sin x + 2}\right| + C$$

Numerical Integration
When we have limits and the integral can't be done algebraically.

Examples of integrals with no analytical antiderivative:

$$\int e^{x^2} dx\, , \int \sqrt{x^3 + 1} dx\, , \int \sin(\ln x)\, dx\, , \int e^x \ln x\, dx$$

Used with definite integrals, and may be used for functions that cannot be integrated by other means, or to estimate complex integrals that be done by other means if we spent the time on them.

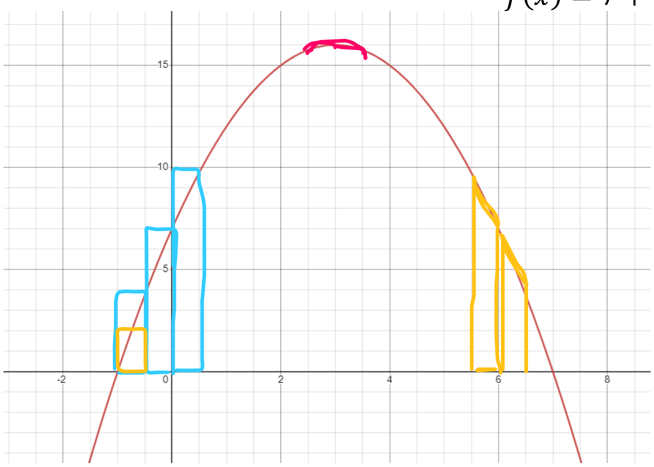
Trapezoidal Rule and Simpson's Rule

Trapezoidal Rule is very easy to explain what is going on and derive the formula easily. Simpson's is more complex to prove, but it is more accurate with fewer steps.

Trapezoidal Rule

When we were doing Reimann sums (back when we started integration), we estimate the area under a curve by using rectangles.

$$f(x) = 7 + 6x - x^2$$



Find the area under the curve f(x).

In Riemann sums we used Area = $f(x_i)\Delta x$

In the Trapezoidal Rule, Area is equal to $\frac{f(x_i)+f(x_{i+1})}{2}\Delta x$ $(A_{trap} = \frac{1}{2}(b_1 + b_2)h$

$$\frac{\Delta x}{2}[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n)] =$$

$$\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \approx \int_a^b f(x)dx$$

$$\Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)dx \approx \frac{\text{b} - \text{a}}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Find the area under the curve $\int_1^3 \sqrt{x}dx$ using $n = 5$.

$$\Delta x = \frac{3-1}{5} = \frac{2}{5} = 0.4$$

$$x_0 = 1, x_1 = 1.4, x_2 = 1.8, x_3 = 2.2, x_4 = 2.6, x_5 = 3$$

$$\int_1^3 \sqrt{x}dx \approx \frac{3-1}{2(5)}\left[\sqrt{1} + 2\sqrt{1.4} + 2\sqrt{1.8} + 2\sqrt{2.2} + 2\sqrt{2.6} + \sqrt{3}\right] \approx \frac{1}{5}[13.973\ldots] \approx 2.7946\ldots$$

Compare to the real answer:

$$\int_1^3 x^{\frac{1}{2}}dx = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^3 = \frac{2}{3}\left[3^{\frac{3}{2}} - 1\right] \approx 2.7974\ldots$$

Simpson's Rule

Estimating the curve with a quadratic function.

$$\int_a^b f(x)dx \approx$$

$$\frac{\mathrm{b}-\mathrm{a}}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

The n you use for this algorithm must be even.

Use n=4 and Simpson's Rule to estimate the area under the curve $f(x) = \sqrt{x}$ on the interval [1,3].

$$\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3$$

$$\int_1^3 \sqrt{x}dx \approx \frac{3-1}{3(4)}\left[\sqrt{1} + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + \sqrt{3}\right] \approx \frac{1}{6}[16.784\ldots] \approx 2.79733\ldots$$

Errors on Trapezoidal Rule and Simpson's Rule

$$E_T \leq \frac{\max\limits_{x\in[a,b]}|f''(x)|\,(b-a)^3}{12n^2}$$

$$E_S \leq \frac{\max\limits_{x\in[a,b]}|f^{IV}(x)|\,(b-a)^5}{180n^4}$$

Suppose I want to estimate the error on $f(x) = \sqrt{x}$ on [1,3] with n=4. Find the estimated error for both Trapezoidal Rule and Simpson's Rule.

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$
$$f^{IV}(x) = -\frac{15}{16}x^{-\frac{7}{2}}$$

$$E_T \leq \frac{\max\limits_{x\in[1,3]}\left|-\frac{1}{4}x^{-\frac{3}{2}}\right|(3-1)^3}{12(4)^2} = \frac{\left(\frac{1}{4}\right)(8)}{12 \times 16} = \frac{1}{96} \approx 0.0104 \ldots$$

$$E_S \leq \frac{\max\limits_{x\in[a,b]}\left|-\frac{15}{16}x^{-\frac{7}{2}}\right|(3-1)^5}{180(4)^4} = \frac{\left(\frac{15}{16}\right)(32)}{180 \times 256} \approx 6.5 \times 10^{-4}$$

Finding the value of n to obtain a particular error.

$$E \leq 10^{-5}$$

$$10^{-5} = E_T \leq \frac{\max\limits_{x\in[1,3]}\left|-\frac{1}{4}x^{-\frac{3}{2}}\right|(3-1)^3}{12n^2}$$

$$10^{-5} \approx \frac{\left(\frac{1}{4}\right)(8)}{12n^2}$$

$$n^2 = \frac{\left(\frac{1}{4}\right)(8)}{12} \times 10^5 \approx 16666.66666 \ldots.$$

$$n \approx 129.0999 \ldots$$
$$n \geq 129.0999 \ldots$$

Set n=130.
Always round up.

$$10^{-5} = E_S \leq \frac{\max\limits_{x\in[1,3]}\left|-\frac{15}{16}x^{-\frac{7}{2}}\right|(3-1)^5}{180n^4}$$

$$n^4 \geq \frac{\left(\frac{15}{16}\right)(32)}{180} \times 10^5 \approx 16666.6666 \ldots.$$

$$n \geq 11.3621 \ldots$$

Set n=12 (round up to the next even integer).