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Partial Fractions
Overview of Integration Techniques

Rational Function integration:
Is the rational function proper or improper: is the numerator a smaller degree than the denominator?

If the numerator is the same degree or larger than the denominator, you do need to do long division first.

Once it's proper, then we can consider how complex it is:

1) Does the denominator factor?
2) Can we use u-sub or another integration rule like arctangent? (may need to complete the square)

If the denominator does factor, then partial fractions is appropriate.

Partial fractions deals with complex rational functions that are proper and whose denominator can be factored, either into linear factors or unfactorable quadratics or a combination of these.

The goal of partial is to separate out our expression into multiple simpler terms that are algebraically equivalent to the original. We are trying to undo the process of finding a common denominator.

$$\frac{2}{x+1}+\frac{4}{x-3}=\frac{2(x-3)}{(x+1)(x-3)}+\frac{4(x+1)}{(x+1)(x-3)}=\frac{6x-2}{(x+1)(x-3)}$$

$$\frac{6x-2}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}$$

$$\frac{A(x-3)}{(x+1)(x-3)}+\frac{B(x+1)}{(x+1)(x-3)}=\frac{Ax-3A+Bx+B}{(x+1)(x-3)}$$

$$Ax-3A+Bx+B=6x-2$$

$$Ax+Bx=6x$$
$$A+B=6$$
$$-3A+B=-2$$

$$A+B=6$$
$$3A-B=2$$

$$4A=8$$
$$A=2$$
$$2+B=6$$
$$B=4$$

$$\int \frac{6x-2}{(x+1)(x-3)}dx = \int \frac{2}{x+1} + \frac{4}{x-3}dx = 2\ln|x+1| + 4\ln|x-3| + C$$

1) Linear factors: The numerator of our guess is constant, $\frac{A}{x-c}$
2) Quadratic factors: the numerator of our guess is linear: $\frac{Ax+B}{x^2+a^2}$

If an individual factor is repeated, then the numerator form is the same for all the terms, but the denominators will take different powers up to the number of repetitions.

$$\frac{p(x)}{(x-3)^3} \to \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$$

Algebraically equivalent to $\frac{ax^2+bx+c}{(x-3)^3}$, but the problem here is that I can't integrate it like this. So setting up like the above takes care of the extra algebra we would need.

$$\frac{p(x)}{(x^2+4)^3} \to \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3}$$

For example, applying all the rules together:

Suppose you needed to decompose the expression $\frac{x^4+3x^2-x^2+x-7}{(x+1)(x-2)^2(x^2+1)(x^2+4)^2}$?

$$\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{x^2+4} + \frac{Hx+I}{(x^2+4)^2}$$

Example.

$$\int \frac{5}{(x+1)(x-2)}dx$$

$$\frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2)}{(x+1)(x-2)} + \frac{B(x+1)}{(x+1)(x-2)}$$

$$Ax - 2A + Bx + B = 5$$

$$A + B = 0$$
$$-2A + B = 5$$

$$A = -B$$
$$2B + B = 5$$
$$3B = 5$$
$$B = \frac{5}{3}, A = -\frac{5}{3}$$

$$\int \frac{\left(-\frac{5}{3}\right)}{x+1} + \frac{\frac{5}{3}}{x-2}\, dx = -\frac{5}{3}\ln|x+1| + \frac{5}{3}\ln|x-2| + C$$

Example.

$$\int \frac{x+2}{x^3+4x^2+x+4}\, dx = \int \frac{x+2}{x^2(x+4)+1(x+4)}\, dx = \int \frac{x+2}{(x^2+1)(x+4)}\, dx$$

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+4} = \frac{(Ax+B)(x+4)}{(x+4)(x^2+1)} + \frac{C(x^2+1)}{(x+4)(x^2+1)} = \frac{Ax^2+4Ax+Bx+4B+Cx^2+C}{(x+4)(x^2+1)}$$

$$Ax^2+4Ax+Bx+4B+Cx^2+C = x+2$$

$$A+C=0$$
$$4A+B=1$$
$$4B+C=2$$

$$A=-C$$
$$-4C+B=1$$
$$C+4B=2$$

$$-4C+B=1$$
$$4C+16B=8$$

$$17B=9$$
$$B=\frac{9}{17}$$
$$4\left(\frac{9}{17}\right)+C=2$$

$$C=-\frac{2}{17}$$

$$A=\frac{2}{17}$$

$$\int \frac{\left(\frac{2}{17}\right)x+\frac{9}{17}}{x^2+1} + \frac{\left(-\frac{2}{17}\right)}{x+4}\, dx = \frac{2}{17}\int \frac{x}{x^2+1}\, dx + \frac{9}{17}\int \frac{1}{x^2+1}\, dx - \frac{2}{17}\int \frac{1}{x+4}\, dx =$$

$$\frac{2}{17}\left(\frac{1}{2}\right)\ln|x^2+1| + \frac{9}{17}\arctan x - \frac{2}{17}\ln|x+4| + C$$

If the unfactorable quadratic has three terms: complete the square, at least for the one with the linear term above it, to do the arctan. You may need to borrow or do some algebra for the x term.

$$\frac{x+2}{x^2+2x+5}$$

$$u = x^2+2x+5, du = 2x+2\, dx = 2(x+1)dx$$

$$\frac{x+1}{x^2+2x+5}+\frac{1}{x^2+2x+5}=\frac{x+1}{x^2+2x+5}+\frac{1}{(x+1)^2+4}$$

Example.

$$\int \frac{x^2+2}{(x^2+1)(x-1)^2}\,dx$$

$$\frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{(x-1)^2}=\frac{(Ax+B)(x-1)^2}{(x^2+1)(x-1)^2}+\frac{C(x^2+1)(x-1)}{(x^2+1)(x-1)^2}+\frac{D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$\frac{(Ax+B)(x^2-2x+1)+C(x^3-x^2+x-1)+D(x^2+1)}{(x^2+1)(x-1)^2}=$$

$$\frac{Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+Cx^3-Cx^2+Cx-C+Dx^2+D}{(x^2+1)(x-1)^2}$$

$$Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+Cx^3-Cx^2+Cx-C+Dx^2+D=x^2+2$$

$$A+C=0\rightarrow A=-C$$
$$-2A+B-C+D=1$$
$$A-2B+C=0$$
$$B-C+D=2$$

$$B+C+D=1$$
$$-2B=0\rightarrow B=0$$
$$B-C+D=2$$

$$C+D=1$$
$$-C+D=2$$

$$2D=3\rightarrow D=\frac{3}{2}$$
$$C+\frac{3}{2}=1$$

$$C=-\frac{1}{2}$$
$$A=\frac{1}{2}$$

$$\int \frac{\frac{1}{2}x}{x^2+1}+\frac{-\frac{1}{2}}{x-1}+\frac{\frac{3}{2}}{(x-1)^2}\,dx=\frac{1}{4}\ln|x^2+1|-\frac{1}{2}\ln|x-1|-\frac{3}{2}\left(\frac{1}{x-1}\right)+C$$

Which Rule do I use?

The rules we've learned so far:

1) Basic Integration Rules
2) U-substitution (change of variables)
3) Integration by parts
4) Trig Substitution
5) Partial Fractions

Compare:

$$\int xe^{x^2}dx \ vs. \int xe^{x}dx$$

The first one is u-sub, the second one is integration by parts.

$$\int x\sqrt{x-1}dx \ vs. \int x\sqrt{x^2-1}dx$$

The first one can be done by change of variables (with $u = \sqrt{x-1}$), or it can be done by integration by parts. The second one could be done by trig sub (because of the square under the radical), but you can use u-sub here.

$$\int x^2\sqrt{x^2-1}dx$$

Probably want to do trig sub. This is not a u-sub.

$$\int \frac{1}{x^2+1}dx \ \ vs. \int \frac{x}{x^2+1}dx \ vs. \int \frac{x^2}{x^2+1}dx$$

The first one is arctangent, the second is u-sub, and the third one is long division.

$$\int \frac{1}{x^2+7x+12}dx \ vs. \int \frac{1}{x^2+7x+16}dx$$

The first case is factor and apply partial fractions. And in the second, complete the square and inverse tangent function.