2/1/2024

Integration by Parts
Trigonometric Integrals
(in online book, 3.1 and 3.2; no class on Tuesday)

Integration by Parts

Is used to "undo" the product rule. Used to integrate the product of two functions when they are not related by a chain rule.

$$\int u dv = uv - \int v du$$

$$u(x) = u$$
$$v(x) = v$$
$$dv = v'(x)dx$$
$$du = u'(x)dx$$

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

How do we get this from the product rule:

$$(uv)' = u'v + v'u$$

$$u(v') = (uv)' - u'v$$

Take the antiderivative of both sides

$$\int u(v') = \int (uv)' - \int u'v$$

$$\int u(v') = uv - \int u'v$$

Rule of thumb:
LIATE:
Logarithms, Inverse (Trig Functions), Algebraic, Trig, Exponential

Choose u by starting on the left end of this list and going rightward.

Logs and Inverse Trig functions do not have nice, easy antiderivative rules.
We can take derivatives of them, however, so this is a good choice for u, which we will take a derivative of.

Algebraic – level A: polynomial functions, level B: rational or radical functions

Trig: usually refers to sine and cosine

Ideally for dv, you want something that will not get more complicated when you integrate (by itself).

Example.
Integrate $\int x \cos x \, dx$

$$u = x, dv = \cos x \, dx$$
$$du = dx, v = \sin x$$

$$\int udv = uv - \int vdu$$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$$

What is the derivative of $(x \sin x + \cos x)$? $\sin x + x \cos x - \sin x = x \cos x$

Example.
Integrate $\int \arcsin x \, dx$

$$u = \arcsin x \, , dv = dx$$
$$du = \frac{1}{\sqrt{1 - x^2}} dx, v = x$$

$$x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx = x \arcsin x + \sqrt{1 - x^2} + C$$

$$w = 1 - x^2, dw = -2xdx \to -\frac{1}{2} dw = xdx$$

$$\int \frac{x}{\sqrt{1 - x^2}} dx = -\frac{1}{2} \int \frac{1}{\sqrt{w}} dw = -\frac{1}{2} \int w^{-\frac{1}{2}} dw = -w^{\frac{1}{2}} = -\sqrt{w} = -\sqrt{1 - x^2}$$

Example.
Integrate $\int x^3 e^{x^2} dx$

$$u = x^2, dv = xe^{x^2} dx$$
$$du = 2xdx, v = \frac{1}{2} e^{x^2}$$

$$w = x^2, dw = 2xdx \to \frac{1}{2} dw = xdx$$

$$\int xe^{x^2} dx = \frac{1}{2} \int e^w dw = \frac{1}{2} e^w = \frac{1}{2} e^{x^2}$$

$$\frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx = \frac{1}{2}x^2e^x - \frac{1}{2}e^{x^2} + C$$

The powers need to be odd if the composite function contains x^2

Multiple integrations by parts:

Integrate $\int x^2e^xdx$

$$u = x^2, dv = e^xdx$$
$$du = 2xdx, v = e^x$$

$$x^2e^x - \int 2xe^xdx$$

$$u = 2x, dv = e^xdx$$
$$du = 2dx, v = e^x$$

$$x^2e^x - \left[2xe^x - \int 2e^xdx\right] = x^2e^x - 2xe^x + \int 2e^xdx = x^2e^x - 2xe^x + 2e^x + C$$

Integrate $\int x^5e^{2x}dx$

Rules for the tabular method:
The u function should be a polynomial (take the derivative until it disappears).
The dv function must be integrable by itself (no borrowing or simplifying with the other function)

+/-	u	dv
+	x^5	e^{2x}
-	$5x^4$	$\frac{1}{2}e^{2x}$
+	$20x^3$	$\frac{1}{4}e^{2x}$
-	$60x^2$	$\frac{1}{8}e^{2x}$
+	$120x$	$\frac{1}{16}e^{2x}$
-	120	$\frac{1}{32}e^{2x}$
+	0	$\frac{1}{64}e^{2x}$

$$\frac{1}{2}x^5e^{2x} - \frac{5}{4}x^4e^{2x} + \frac{20}{8}x^3e^{2x} - \frac{60}{16}x^2e^{2x} + \frac{120}{32}xe^{2x} - \frac{120}{64}e^{2x} + C$$

Change of variables vs. integration by parts

Integrate $\int x\sqrt{x+1}dx$

Version 1: Change of variables

$$u=\sqrt{x+1}, u^2=x+1 \to x=u^2-1, dx=2udu$$

$$\int (u^2-1)u2udu=\int 2u^4-2u^2du=\frac{2}{5}u^5-\frac{2}{3}u^3+C \to \frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+C$$

Version 2: Integration by parts

$$u=x, dv=(x+1)^{\frac{1}{2}}dx$$
$$du=dx, v=\frac{2}{3}(x+1)^{\frac{3}{2}}$$

$$\frac{2}{3}x(x+1)^{\frac{3}{2}}-\int \frac{2}{3}(x+1)^{\frac{3}{2}}dx=\frac{2}{3}x(x+1)^{\frac{3}{2}}-\frac{2}{3}\left(\frac{2}{5}\right)(x+1)^{\frac{5}{2}}+C$$

$$\frac{2}{3}(x+1)^{\frac{3}{2}}\left[x-\frac{2}{5}(x+1)\right]+C$$

Looping integrals
Integrate $\int e^x \sin x\, dx$

$$u=\sin x, dv=e^x dx$$
$$du=\cos x\, dx, v=e^x$$

$$e^x \sin x-\int e^x \cos x\, dx$$

$$u=\cos x, dv=e^x dx$$
$$du=-\sin x\, dx, v=e^x$$

$$e^x \sin x-\left[e^x \cos x-\int -e^x \sin x\, dx\right]$$

$$\int e^x \sin x\, dx=e^x \sin x-e^x \cos x-\int e^x \sin x\, dx$$

Add the common integrals together on the left

$$2\int e^x \sin x\, dx=e^x \sin x-e^x \cos x$$
$$\int e^x \sin x\, dx=\frac{1}{2}e^x \sin x-\frac{1}{2}e^x \cos x+C$$

Trigonometric Integrals
Powers of trig functions in combination with each other.

Sine/cosine integrals, tangent/secant integrals, (cotangent/cosecant integrals), other combinations

When dealing with sine and cosine integrals.

$$\int \sin^m x \cos^n x \, dx$$

1) If m is odd, pull out one sine to be the du and convert the remaining sines to cosines, let u=cos x
2) If n is odd, pull out one cosine to be the du, and convert the remaining cosines to sines, let u=sinx
3) If both are even, use the power-reducing identities (possibly repeatedly) until you have only linear terms in sine and cosine

Example.

$$\int \sin^3 x \cos^4 x \, dx$$

$$\int \sin^2 x \cos^4 x \, (\sin x \, dx)$$

$$\int (1 - \cos^2 x) \cos^4 x \, (\sin x \, dx)$$

$$u = \cos x \,, du = -\sin x \, dx$$

$$-\int (1 - u^2) u^4 du = -\int u^4 - u^6 du = -\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right] + C = -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

Example.

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x)dx = \frac{1}{4}\int 1 - \cos^2 2x \, dx =$$

$$\frac{1}{4}\int 1 - \frac{1}{2}(1 + \cos 4x)dx = \frac{1}{4}\int 1 - \frac{1}{2} - \frac{1}{2}\cos 4x \, dx = \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x \, dx = \frac{1}{8}\int 1 - \cos 4x \, dx =$$

$$\frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + C$$

Tangent/Secant

$$\int \tan^m x \sec^{\mathrm{n}} x \, dx$$

1) If n is even, then pull out two copies of secant and convert the remaining secants to tangents, let u=tan x
2) If m is odd, then pull out one tangent and one secant, convert the remaining tangents to secants, and let u=sec x
3) If n is odd, and m is even, then integration by parts (and the integral loops… no fun at all!)

Other cases:
Convert everything to sine and cosine and try that… if all else fails.