1/25/2024

Work
Probability Applications

Work

Force times distance = work in the simplest cases. But what if the force or the distance is changing?

Spring problems:
Hook's Law: $F = kx$

x is the distance the spring is stretched from equilibrium

Suppose that we know that 16 N of force can stretch a spring 10 cm. Find the work done stretching the spring an additional 10 cm.

N goes with meters
Pounds goes with feet

$$x = 0.1\ m$$
$$F = 16 = k(0.1)$$
$$k = 160$$

Work:

$$W = \int_a^b F(x)dx = \int_{0.1}^{0.2} 160x dx = 80x^2|_{0.1}^{0.2} = 80(0.2^2 - 0.1^2) = 80(0.3) = 2.4\ Nm$$

If you need to convert mass to a force: $F = ma$

There are other kinds of problems that work similarly to this example.

Gravity problems, force between charged particles.

$$F = \frac{Gm_1m_2}{d^2} = \frac{k}{x^2}$$

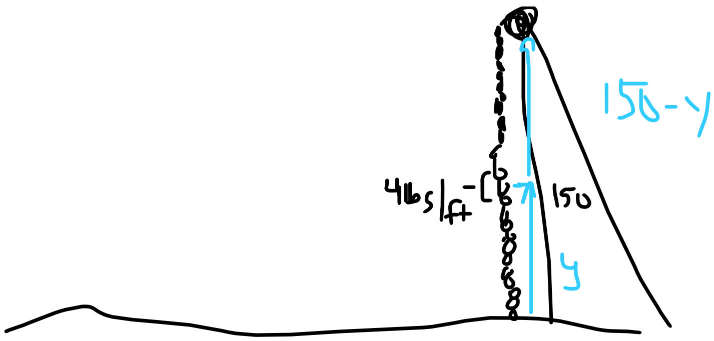
Suppose you want to launch a rocket into orbit 100 miles above the surface. (Treat the radius of the earth as approximately 4000 miles). Find the work done putting the rocket into orbit. The rocket weighs 10 tons on the surface of the Earth.

$$W = \int_{4000}^{4100} \frac{1.6 \times 10^8}{x^2} dx$$

$$10 = \frac{k}{4000^2}$$
$$k = 1.6 \times 10^8$$

Chain problems

Suppose that a chain that weighs 4 pounds per foot is hanging from a crane that is 150 feet tall down to the ground. Find the work done in winding up the whole chain.



$$F = density \times unit\ of\ length = 4\frac{lbs}{ft}dy\ (feet)$$

An equation for the distance given the starting position of the segment

$$distance\ traveled = 150 - y$$

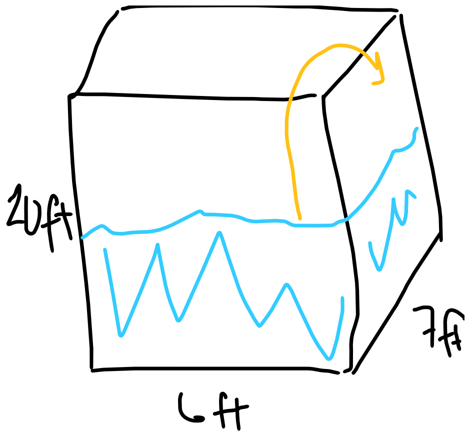
$$W = \int_a^b 4(150 - y)\ dy$$

If I'm winding the chain all the way to top and the chain is hanging all the way to the ground then the limits are 0 and 150

If you are winding the chain to only half wound up then the upper limit would be 75.
If the chain starts only hanging down halfway, then you'd start at 75, then the upper limit would be 150.

$$W = 4\int_0^{150} 150 - y\ dy = 4\left[150y - \frac{1}{2}y^2\right]_0^{150} = 4[22{,}500 - 11{,}250] = 4[11{,}250]$$
$$= 45{,}000\ foot - pounds$$

Tank problem
Suppose we have a rectangular tank that is 6 feet wide by 7 feet and 20 feet deep.
The tank is half full of water and we want to find the work done pumping the remaining water out of the tank over the top rim.



Density of water in imperial units is 62.4 lbs/ft^3, in SI force is 9800 N/m^3

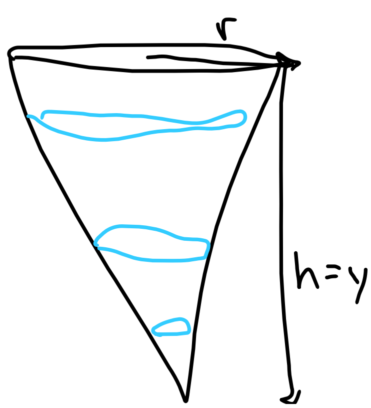
$$Force = weight = density \times volume = density \times area \times dy = density \times 6 \times 7 \times dy = 2620.8\, dy$$

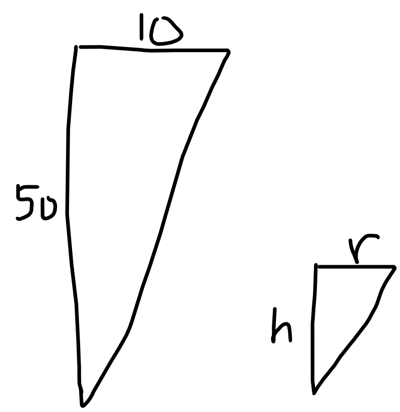
Distance moved: how high up from the bottom did you start (y) and how far is there to go? Top minus y

$$distance = 20 - y$$

$$W = \int_0^{10} 2620.8(20-y)dy = 2620.8\left[20y - \frac{1}{2}y^2\right]_0^{10} = 2620.8[150] = 293{,}120\ foot - pounds$$

Conical tank





The relationship between the height and the radius of the cross section depends on similar triangles.

Suppose we have a conical tank (point down) that is full of water. The tank has a depth of 50 m, and a radius at the top of 10 m. Find the work done pumping all the water out over the top of the tank.

$$\frac{50}{h} = \frac{10}{r}$$

$$\frac{50}{y} = \frac{10}{r}$$

Solve for r in terms of y.

$$10y = 50r$$
$$\frac{1}{5}y = r$$

$$Force = weight = density \times volume = density \times area \times dy =$$
$$density \times \pi r^2 \times dy = density \times \pi \times \left(\frac{1}{5}y\right)^2 \times dy$$

Distance traveled: top of the tank minus y

$$W = \int_0^{50} \frac{9800\pi}{25} y^2(50 - y)dy = \frac{9800\pi}{25} \int_0^{50} 50y^2 - y^3 dy =$$

$$\frac{9800\pi}{25}\left[\frac{50}{3}y^3 - \frac{1}{4}y^4\right]_0^{50} = \frac{9800\pi}{25}\left[\frac{1562500}{3}\right] = \frac{612500000}{3}\pi\ Nm$$

Hemispherical tank.
Widest distance is the top

Radius at the widest point is 40
The radius of any slice is $r = x = \sqrt{40^2 - y^2}$

Probability Applications

The probability density function defines a probability curve if the area under the curve is equal to 1.

Suppose a probability density function in one variable is given by $f(x) = Kx^2, 0 \le x \le 4$

$$1 = \int_0^4 Kx^2 dx = \frac{K}{3}x^3|_0^4 = \frac{K}{3}[64]$$

$$K = \frac{3}{64}$$

Given $f(x) = \frac{3}{64}x^2, 0 \le x \le 4$

Find the probability that x is between 1 and 2: $P(1 \le x \le 2)$

$$P = \int_1^2 \frac{3}{64}x^2 dx = \frac{3}{64}\left[\frac{1}{3}x^3\right]_1^2 = \frac{3}{64}\left(\frac{1}{3}\right)[8 - 1] = \frac{7}{64}$$

Find the mean of the distribution:

$$mean = \int_0^4 xf(x)dx = \int_0^4 \frac{3}{64}x^3 dx = \frac{3}{64}\left[\frac{1}{4}x^4\right]_0^4 = 3$$

Variance = $\int_a^b (x - mean)^2 f(x)dx$

$$V = \int_0^4 \frac{3(x-3)^2}{64} x^2 dx$$

Standard deviation is the square root of the variance.

Median is the point where the area below that point is 50% (50% of the data is lower than that number and 50% is higher)

$$0.5 = \int_0^t \frac{3}{64} x^2 dx$$

You can do this same trick for any percentile. 90^{th} percentile, replace 0.5 in the above with 0.9.