1/23/2024

Arclength and surface area of revolution (2.4)

From the arclength handout, we found that

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

Linear example.
Find the length of arc along the function $y = 3x$ on the interval [0,3]

$$s = \int_0^3 \sqrt{1 + (3)^2}dx = \int_0^3 \sqrt{1 + 9}\, dx = \int_0^3 \sqrt{10}dx = 3\sqrt{10}$$

Starting point at x=0 is (0,0), and the ending point at x=3, (3,9)

$$d = \sqrt{(3-0)^2 + (9-0)^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$$

Some functions that will work well algebraically with the arclength formula:
Some hyperbolic cosine functions (circle: $x^2 + y^2 = 1$, but hyperbolic: $x^2 - y^2 = 1$)

$$\cosh^2 x - \sinh^2 x = 1$$

Instead of

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Another example that works well is $y = x^{\frac{3}{2}}$
Another example that works well is $y = \ln(\cos x)$
Another class of functions that work well has a form with a polynomial, and rational term. So that when you take the derivative you get x^n for the polynomial and x^-n for the rational term (give or take a constant)

Find the length of arc of the function $y = \cosh x$ from [0,1].

$$y' = \sinh x$$

$$s = \int_0^1 \sqrt{1+\sinh^2 x}\, dx = \int_0^1 \sqrt{\cosh^2 x}\, dx = \int_0^1 \cosh x\, dx = \sinh x \,|_0^1 = \sinh 1$$

$$= \frac{e^1 - e^{-1}}{2} = \frac{1}{2}\left(e - \frac{1}{e}\right)$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh^2 x = 1 + \sinh^2 x$$

Find the length of arc of the function $y = x^{\frac{3}{2}}$ on the interval [1,4]

$$s = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x dx}$$

$$u = 1 + \frac{9}{4}x, du = \frac{9}{4}dx \rightarrow \frac{4}{9}du = dx$$

$$s = \int_{13/4}^{10} \frac{4}{9}u^{\frac{1}{2}}du = \frac{4}{9}\left(\frac{2}{3}\right)u^{\frac{3}{2}}|^{10}{}_{13/4} = \frac{8}{27}\left[10^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}}\right]$$

From textbook, 2.4 #177: $y = \frac{x^4}{4} + \frac{1}{8x^2} = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}$ on the interval [1,2]

$$y' = x^3 - \frac{1}{4}x^{-3}$$

$$s = \int_1^2 \sqrt{1 + \left(x^3 - \frac{1}{4}x^{-3}\right)^2}\, dx = \int_1^2 \sqrt{1 + x^6 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16}x^{-6}dx} =$$

$$\int_1^2 \sqrt{1 + x^6 - \frac{1}{2} + \frac{1}{16}x^{-6}dx} = \int_1^2 \sqrt{x^6 + \frac{1}{2} + \frac{1}{16}x^{-6}dx} = \int_1^2 \sqrt{\left(x^3 + \frac{1}{4}x^{-3}\right)^2}\, dx$$

$$= \int_1^2 x^3 + \frac{1}{4}x^{-3}dx = \frac{1}{4}x^4 - \frac{1}{8x^2}|_1^2 = \frac{1}{4}(16) - \frac{1}{32} - \frac{1}{4} + \frac{1}{8} = \frac{123}{32}$$

Find the length of arc of the function $y = x^3$ on the interval [0,2].

$$s = \int_0^2 \sqrt{1 + (3x^2)^2}\, dx = \int_0^2 \sqrt{1 + 9x^4}dx$$

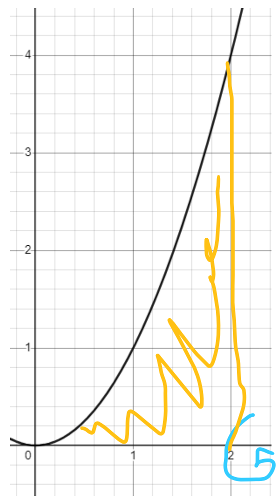
Do it numerically in your calculator: MATH → 9: FnInt
fnInt(function, x, lower limit, upper limit)

fnInt(sqrt(1+9x^4),x,0,2)

$$\approx 8.63032922 \ldots$$

Surfaces of revolution.

Suppose I take the function $y = x^2$ and I rotate it around the x-axis on the interval [0,2]. Find the surface area of revolution.



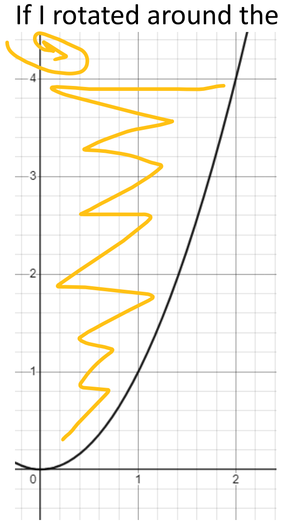
$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx$$

Rotating around the x-axis: the r(x)=f(x), if you are rotating around the y-axis, then r(x)=x

In this example:

$$SA = 2\pi \int_0^2 x^2\sqrt{1 + (2x)^2}dx = 2\pi \int_0^2 x^2\sqrt{1 + 4x^2}dx$$

If I rotated around the y-axis



$$S = 2\pi \int_0^2 x\sqrt{1 + 4x^2}dx$$

This example rotated around the y-axis, this is integrable by hand using u-substitution.

If you rotate $y = x^3$ around the x-axis

$$S = 2\pi \int_a^b x^3\sqrt{1 + 9x^4}dx$$

If you rotate around the y-axis

$$S = 2\pi \int_a^b x\sqrt{1 + 9x^4}dx$$

Here the y-axis one cannot be done (yet), but the x-axis one can be done with regular u-sub.

Next sections: Work, probably applications, centers of mass