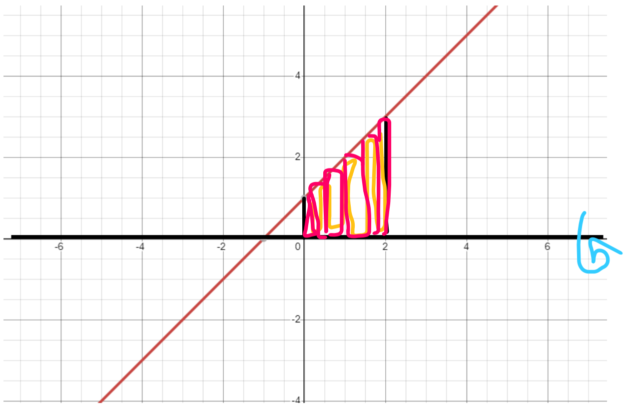
1/18/2024

Volumes of Solids of Revolution:
Disk/Washer Method (volumes by slicing) (2.2 of the online text book)
Shell Method (2.3 of the online text book)

Disk/Washer Method

Suppose we have the function $f(x) = x + 1$, rotate around the x-axis between [0,2].



If I take a little rectangle and I rotate that rectangle around the x-axis, I end up with a cylinder. The radius of that cylinder is the height of the function, and therefore the area of the face is $A = \pi r^2 = \pi[f(x_i)]^2$, Then the volume of the disk is the width of the original rectangle times the area of the face:

$$V = \pi[f(x_i)]^2 \Delta x$$

Total volume estimate: $V = \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$

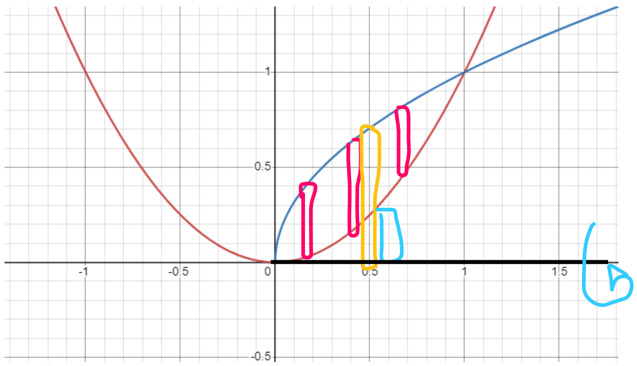
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \int_a^b \pi[f(x)]^2 dx$$

$$\pi \int_0^2 (x+1)^2 dx = \pi \int_0^2 x^2 + 2x + 1\, dx = \pi \left[\frac{1}{3}x^3 + x^2 + x\right]_0^2 = \pi\left[\frac{8}{3} + 4 + 2\right] = \frac{26\pi}{3}$$

Disk method because the region we are rotating touches the axis of rotation.

Washer method:

Find the volume of revolution from rotating the region bounded by $y = x^2$ and $y = \sqrt{x}$ around the x-axis.



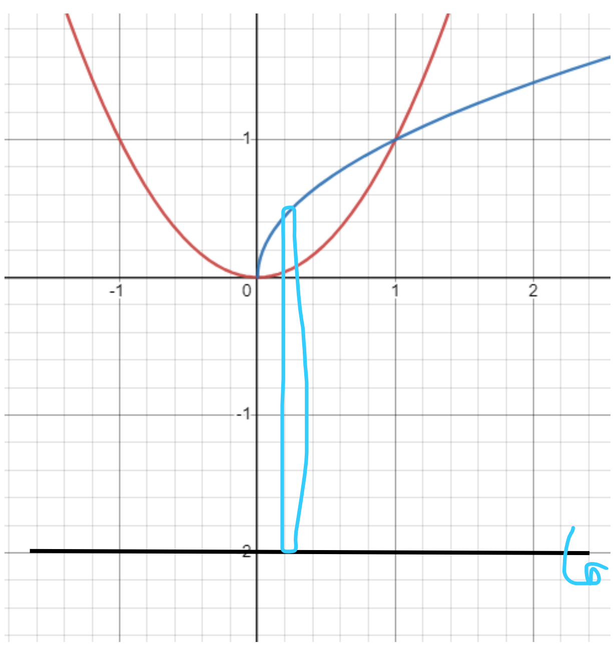
$$V = \pi \int_a^b \left[Radius_{outer}^2 - Radius_{inner}^2\right] dx = \pi \int_a^b [f(x)]^2 dx - \pi \int_a^b [g(x)]^2 dx$$

To find the volume, the outer radius is the square root of x, the inner radius is x^2 function.

$$V = \pi \int_0^1 \left(\sqrt{x}\right)^2 - (x^2)^2 \, dx = \pi \int_0^1 x - x^4 \, dx = \pi \left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3\pi}{10}$$

Rotating around any line parallel to the x-axis.

Find the volume of revolution of the region bounded by $y = \sqrt{x}$ and $y = x^2$ rotated around the line $y = -2$.

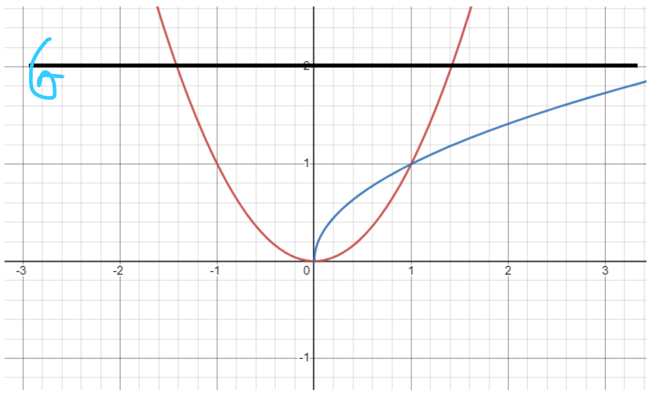


Radius of the outer surface: the function minus the axis of rotation: $R_{outer} = \sqrt{x} - (-2) = \sqrt{x} + 2$
Radius of the inner surface: the function minus the axis of rotation: $R_{inner} = x^2 - (-2) = x^2 + 2$

$$V = \pi \int_0^1 \left(\sqrt{x}+2\right)^2 - (x^2+2)^2\, dx = \pi \int_0^1 x + 4\sqrt{x} + 4 - (x^4 + 4x^2 + 4) dx =$$

$$\pi \int_0^1 x + 4\sqrt{x} - x^4 - 4x^2 dx = \pi \left[\frac{1}{2}x^2 + 4\left(\frac{2}{3}\right)x^{\frac{3}{2}} - \frac{1}{5}x^5 - \frac{4}{3}x^3\right]_0^1 = \pi \left[\frac{1}{2} + \frac{8}{3} - \frac{1}{5} - \frac{4}{3}\right] = \frac{49\pi}{30}$$

If you are rotating around a line parallel to the x-axis but above the region:



Rotate the region around the line $y = 2$ (above the region)
The square root is now the inner radius because it's closer to the line y=2, and the square function is further from the line y=2 so it's the outer radius.

Radius is the axis of rotation minus the function because the axis of rotation here is larger than the function values.

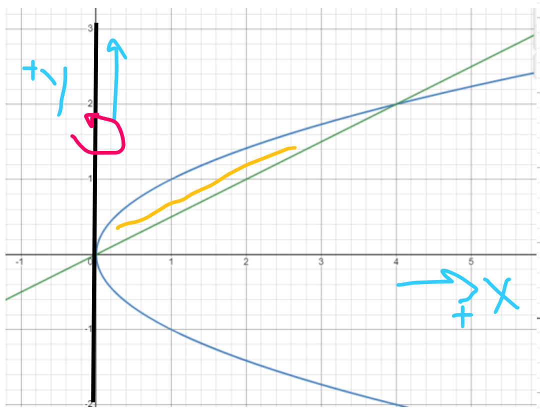
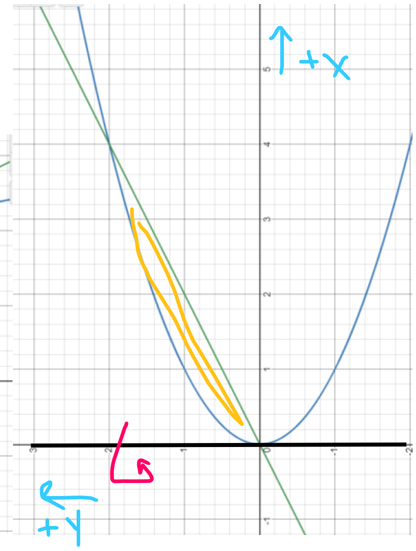
$$V = \pi \int_0^1 (2 - x^2)^2 - \left(2 - \sqrt{x}\right)^2\, dx$$

You can use the disk/washer method to rotate around the y-axis but…

The axis of rotation and the primary axis of rotation need to be the same variable. Functions of x (i.e. f(x)) rotate around the x-axis (or lines parallel to the x-axis) using the disk or washer method.

If I want to use this method to rotate around the y-axis, then I need a function of y, i.e. x=f(y)

Find the volume of the solid of revolution bounded by $x = y^2, x = 2y$ rotated around the y-axis.

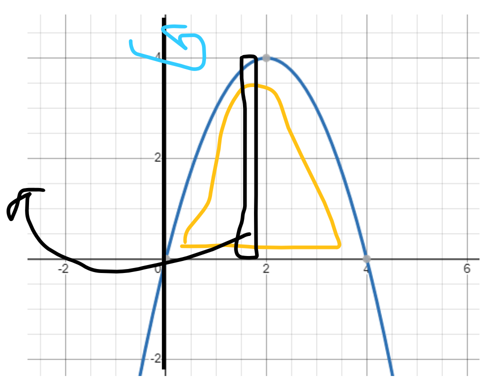
$$V = \pi \int_0^2 (2y)^2 - (y^2)^2 \, dy$$

Washer/Disk: the function and the variable of the axis of rotation need to match. (axis of rotation: we mean the principal axis: is the line parallel to x-axis or the y-axis?)

Shell method.

We have a function of x, and we are rotating around the y-axis (around the opposite axis of the variable).

Find the volume of the solid of revolution bounded by the function $f(x) = 4x - x^2$ and the x-axis, rotated around the y-axis.



Cylinder: height of the function $f(x_i)$, radius of the cylinder: x_i
Fold out the hollow cylinder into a rectangle: height of the rectangle is the height of the cylinder. What is the width of the rectangle: it's the circumference of the circle at the top of the tube: $C = 2\pi r$

Area of the rectangle: $f(x_i)\ (2\pi\ x_i)$
The volume is now the thickness of the rectangle = the thick of the original rectangle that we rotated: Δx

Volume estimate for one shell: $V = 2\pi x_i f(x_i) \Delta x$

$$V_{est} = \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x$$

Volume:

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x = 2\pi \int_a^b x f(x) dx$$

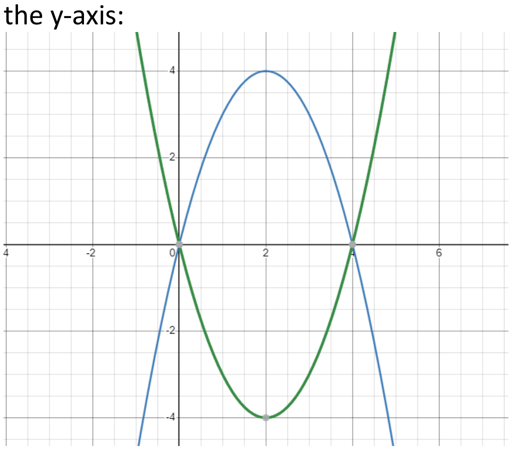
Volume for the example:

$$V = 2\pi \int_0^4 x(4x - x^2)dx = 2\pi \int_0^4 4x^2 - x^3 dx = 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = 2\pi \left[\frac{256}{3} - 64\right] = \frac{128\pi}{3}$$

With a top and bottom function:

$$V = 2\pi \int_a^b x\big(f(x) - g(x)\big)dx$$

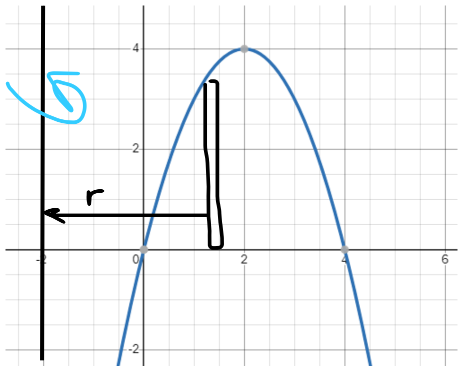
To find the volume of the solid of revolution bounded by $y = 4x - x^2$ and $y = x^2 - 4x$ rotated around the y-axis:



$$V = 2\pi \int_0^4 x[4x - x^2 - (x^2 - 4x)]dx$$

If you are rotating around a line parallel to the y-axis, this does not change the height of the functions, but it does change the radius of shells.

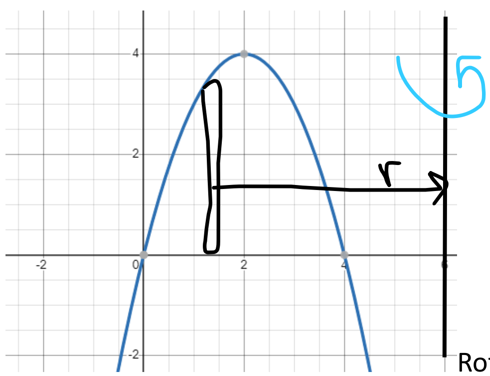
Rotate the region around the line $x = -2$



New radius is the x minus the axis of rotation: $x - (-2) = x + 2$

$$V = 2\pi \int_0^4 (x + 2)(4x - x^2)dx$$

If the new radius is to the right of the region, then the radius becomes the axis of rotation minus x.

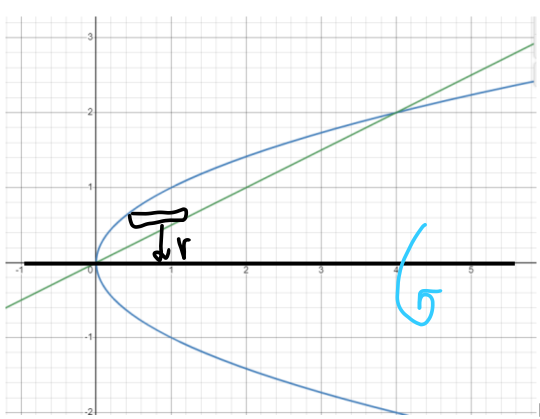


Rotate around x=6

$$V = 2\pi \int_0^4 (6 - x)(4x - x^2)dx$$

Axes of rotation can never be inside the region.

To use the shell method to rotate around the x-axis, but you would need a function of y to do it.



rotate this region around the x-axis.

$$V = 2\pi \int_0^2 y(2y - y^2)dy$$