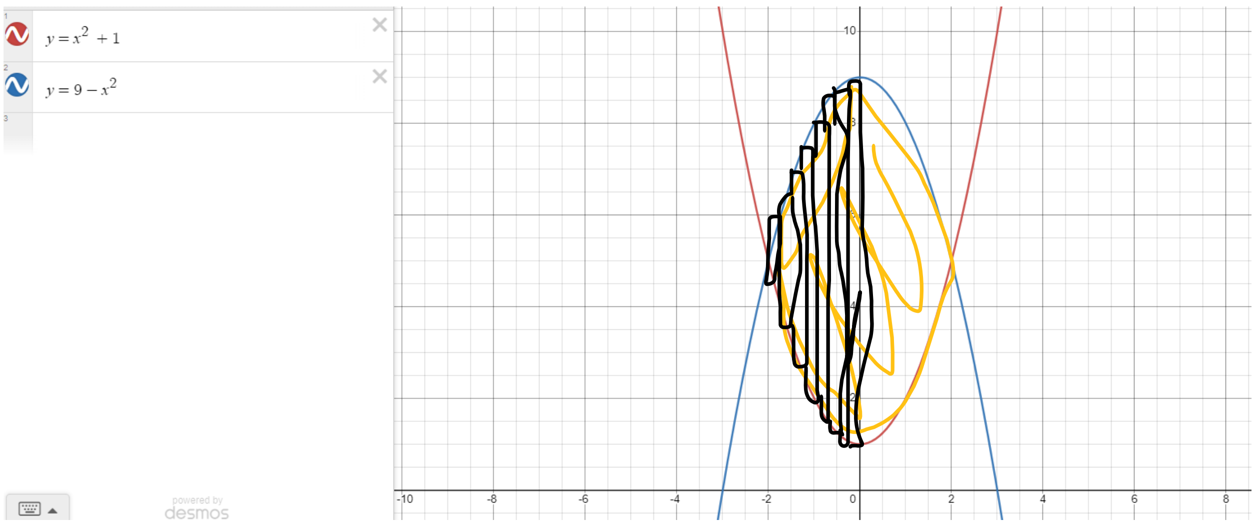
1/16/2024

Introduction to the course
Area between curves

In the online textbook, this is Chapter 2.1.



Suppose we want to find the area between the curves. Note that the intersections are at x= -2, and x=2.

We can build rectangles, like we did for the area under a curve (right-hand, left-hand or midpoint rule). The height of the rectangles is the y-value of the top of the rectangle minus the y-value of the bottom of the rectangle. If the top function is f(x) and the bottom function is g(x), then the height is $f(x_i) - g(x_i)$. Then the total area of the rectangle is the height times Δx, so we get $[f(x_i) - g(x_i)]\Delta x$. Then to get total area, we add all the rectangles.

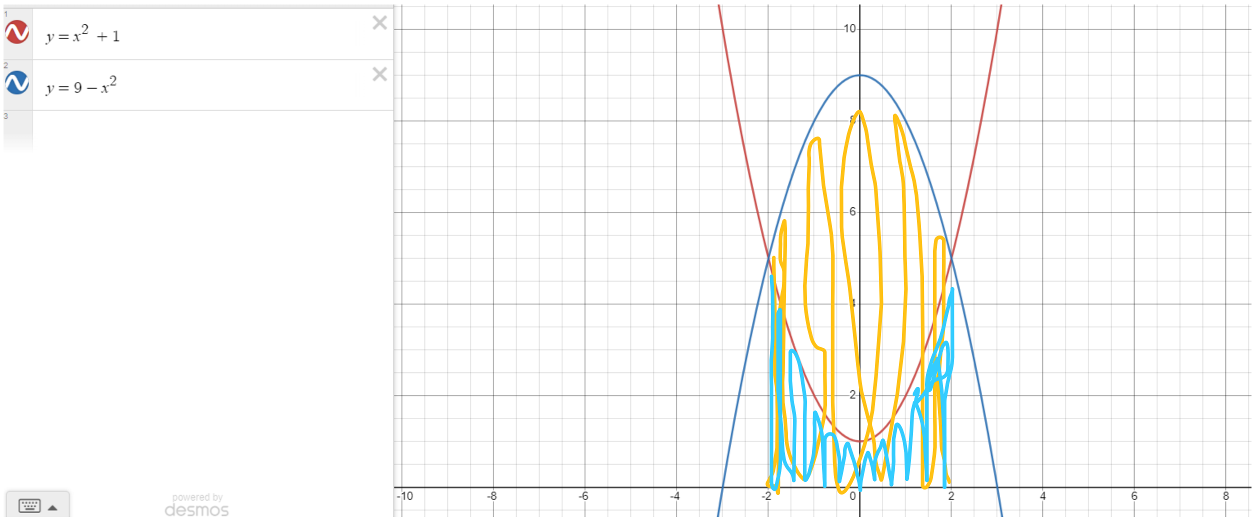
$$A \approx \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x - \sum_{i=1}^{n} g(x_i)\Delta x$$

Let the limit of n go to infinity to get the exact area

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x - \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x$$

$$A = \int_a^b \big(f(x) - g(x)\big)dx \ = \int_a^b f(x)dx - \int_a^a g(x)dx$$

To calculate the area between curves, take the difference of the functions and integrate over the interval.



Consider finding the area under f(x) (top curve) down to the x-axis: $A = \int_a^b f(x)dx$, and then remove the area we don't need under g(x) (bottom curve) down to the x-axis: $A = \int_a^b g(x)dx$. Then subtract them:

Area we want: $\int_a^b f(x)dx - \int_a^b g(x)dx$ = $\int_a^b [f(x) - g(x)]dx$.

A third way to think about this is that we've been doing this the whole time: but we've previously used the special case of g(x)=0. The x-axis is the line y=0.

$$A = \int_a^b f(x) - g(x)dx = \int_a^b [f(x) - 0]dx = \int_a^b f(x)dx$$

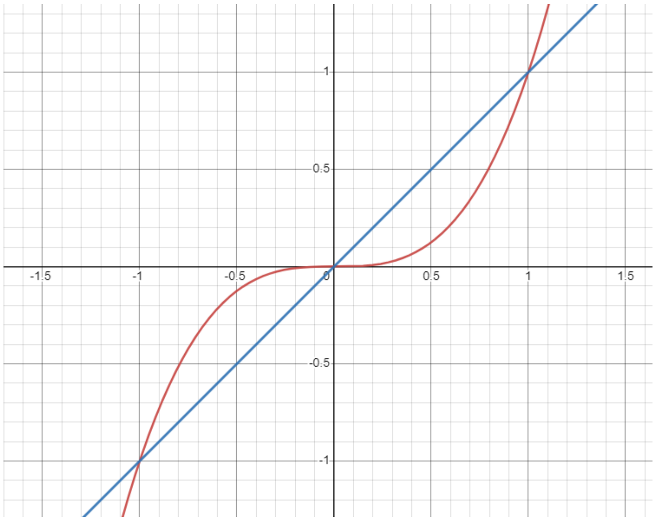
Example.
Find the area between the curves: $g(x) = x^2 + 1$, and $f(x) = 9 - x^2$

$$x^2 + 1 = 9 - x^2$$
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

$$\int_{-2}^{2} 9 - x^2 - (x^2 + 1)dx = \int_{-2}^{2} 8 - 2x^2dx = 2\int_0^2 8 - 2x^2dx = 2\left[8x - \frac{2}{3}x^3\right]_0^2 = 2\left[16 - \frac{16}{3}\right] = \frac{64}{3}$$

Sometimes the problems will give you limits to start with. It always a good idea to graph the functions. If the functions change relationship then you will need to split the integral into two pieces.

Example.
Find the area between the curves $y = x, y = x^3$



Split the integral at the crossover point, x=0, and change the order of subtraction to obtain a positive area for both parts.

$$A = \int_{-1}^{0} x^3 - x\,dx + \int_{0}^{1} x - x^3\,dx$$

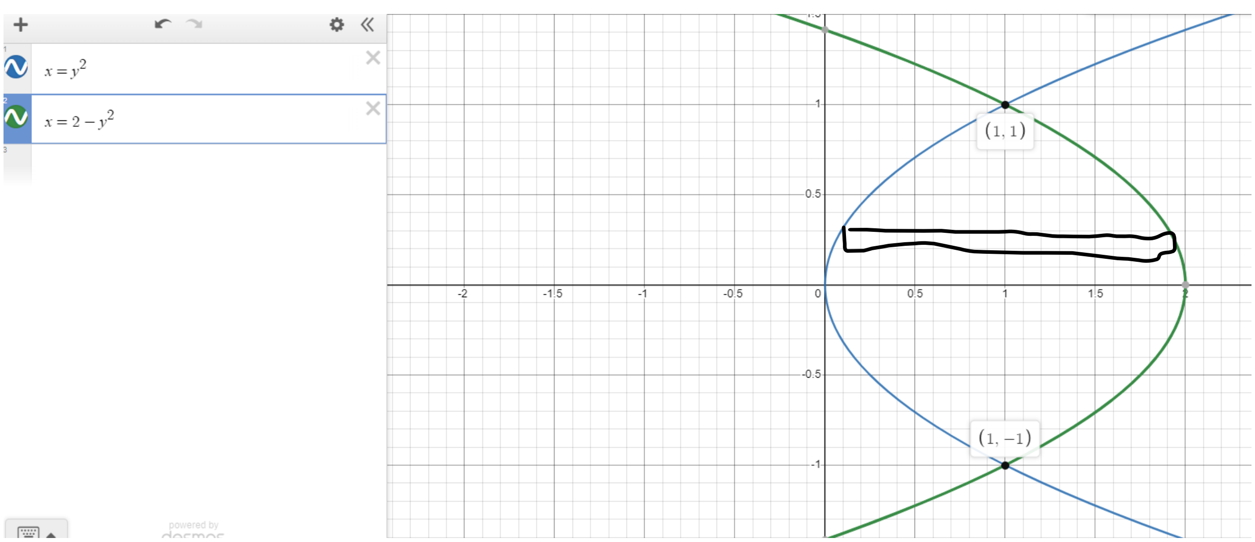
Since these are completely symmetric:

$$= 2\int_{0}^{1} x - x^3\,dx$$

But this will not always be the case.

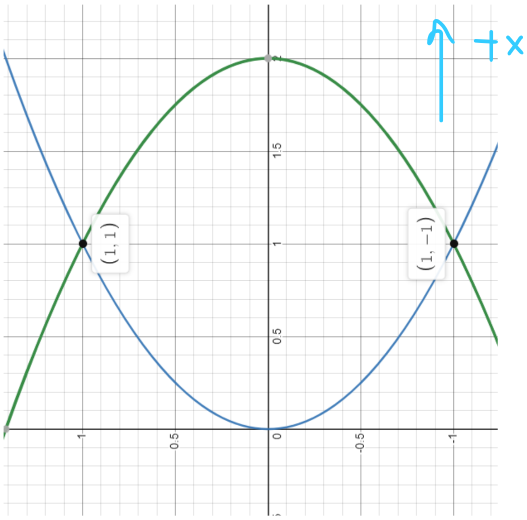
If you have a function of x in terms of y, say $f(y) = x \; and \; g(y) = x$ then the rectangles will need to be horizontal, and the integration will need to be in terms of y.

In the usual orientation, f(x) is the top function, and g(x) is the bottom function.
In this orientation: f(x) is the rightmost function (largest in x) and the g(x) is the leftmost function (smallest in x).



The rightmost function is $f(y) = 2 - y^2, and\ g(y) = y^2$ (leftmost function)

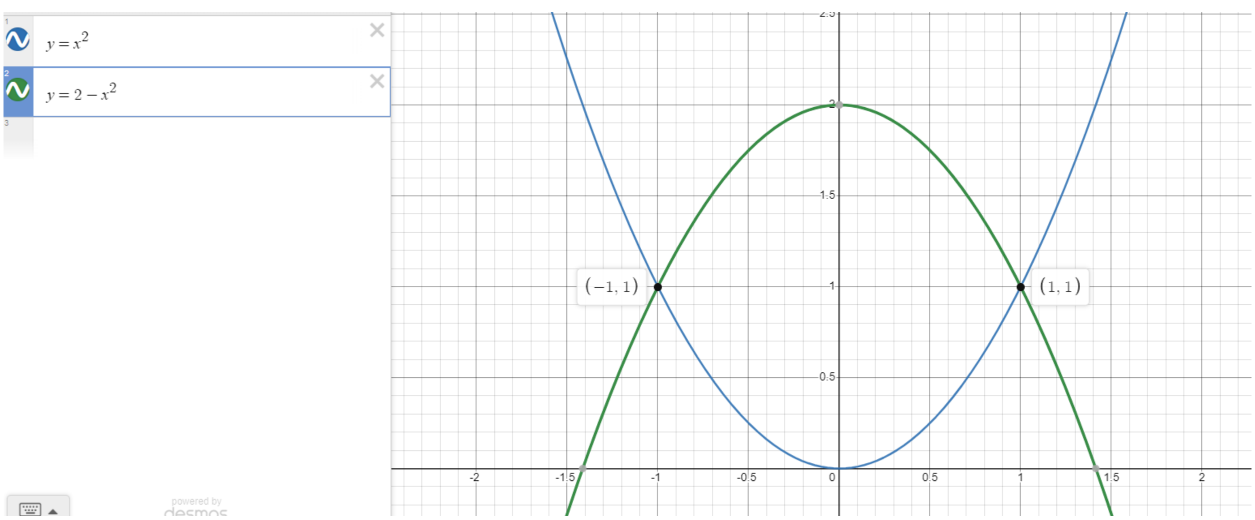
$$A = \int_{-1}^{1} (2 - y^2) - y^2\, dy = \int_{-1}^{1} 2 - 2y^2\, dy = 2\int_{0}^{1} 2 - 2y^2 dy = 2\left[2y - \frac{2}{3}y^3\right]_0^1 = 2\left[2 - \frac{2}{3}\right] = \frac{8}{3}$$



One other approach is to switch all the variables in your problem: replace all y's with x's and all x's with y's.

Consider:
Our initial problem was $f(y) = 2 - y^2, g(y) = y^2$

Switching over: $f(x) = 2 - x^2, g(x) = x^2$



You will get exactly the same area doing this calculation as with the original problem.